Companion forms over totally real fields

Toby Gee *

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Abstract

We show that if F is a totally real field in which p splits completely and f is a mod p Hilbert modular form with parallel weight $2 < k < p$, which is ordinary at all primes dividing p and has tamely ramified Galois representation at all primes dividing p, then there is a "companion form" of parallel weight $k' := p + 1 - k$. This work generalises results of Gross and Coleman–Voloch for modular forms over $\mathbf{Q}$.

1 Introduction

Theorems on "companion forms" were proved by Gross ([Gro90]) under the assumption of some unchecked compatabilities, and then reproved by Coleman and Voloch ([CV92]) without such assumptions. We generalise the methods of Coleman and Voloch to totally real fields.

If $f \in S_k(\Gamma_1(N); \overline{\mathbf{F}}_p)(\epsilon)$ is a mod p cuspidal eigenform, where $p \nmid N$, there is a continuous, odd, semisimple Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_p)$$

attached to f. A famous conjecture of Serre predicts that all continuous odd irreducible mod p representations should arise in this fashion. Furthermore, the "strong Serre conjecture" predicts a minimal weight k_ρ and level N_ρ, in the sense that $\rho \cong \rho_g$ for some eigenform g of weight k_ρ and level N_ρ (prime to p), and if $\rho \cong \rho_f$ for some eigenform f of weight k and level N prime to p then $N_\rho | N$ and $k \geq k_\rho$. The question as to whether all continuous odd irreducible mod p Galois representations are modular in this sense is still open, but the implication "weak Serre $\Rightarrow$ strong Serre" is essentially known (aside from a few cases where $p = 2$).

In solving the problem of weight optimisation it becomes necessary to consider the companion forms problem; that is, the question of when it can occur that we have $f = \sum a_n q^n$ of weight $2 \leq k \leq p$ with $a_p \neq 0$, and an

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eigenform $g = \sum b_n q^n$ of weight $k' = p + 1 - k$ such that $na_n = n^k b_n$ for all n. Serre conjectured that this can occur if and only if the representation ρ_f is tamely ramified above p. This conjecture has been settled in most cases in the papers of Gross ([Gro90]) and Coleman-Voloch ([CV92]).

We generalise these results to the case of parallel weight Hilbert modular forms over totally real fields F. We assume that p splits completely in F; the extension of our results to the general unramified case is work in progress. Our arguments are based on those of [CV92], with several non-trivial and crucial adjustments (see below). Our main theorem is the following:

Theorem A. *Let F be a totally real field in which an odd prime p splits completely. Let π be a mod p Hilbert modular form of parallel weight $2 < k < p$ and level $\mathfrak{n}$, with $\mathfrak{n}$ coprime to p. Suppose that π is ordinary at all primes $\mathfrak{p}|p$, and that the mod p representation $\overline{\rho}_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is irreducible and is tamely ramified at all primes $\mathfrak{p}|p$. Then there is a companion form π' of parallel weight $k' = p + 1 - k$ and level $\mathfrak{n}$ satisfying $\overline{\rho}_{\pi'} \cong \overline{\rho}_\pi \otimes \chi^{k'-1}$.*

In the case $F = \mathbf{Q}$, the mod p Galois representations associated to modular forms may be found in the p-adic Tate modules of the Jacobians of certain modular curves. For other totally real fields one must instead use the Jacquet-Langlands correspondence to realise them in the Tate modules of the Jacobians of Shimura curves associated to certain quaternion algebras (see [Car86b]). Although arguments on "level lowering" for mod p modular forms have been generalised to the totally real case by arguing on these Shimura curves (cf. [Jar99]), it does not seem to be possible to generalise the methods of [Gro90] or [CV92] to work on these curves, because they have no "modular" interpretation as PEL Shimura varieties. Indeed, it is not even clear that there is a natural "Hasse invariant" on the curves corresponding to quaternion algebras. We work instead on certain unitary curves defined over degree two CM extensions E of F, which are of PEL type. In order to base change the Hilbert modular forms to automorphic forms on these Shimura curves it is necessary to twist by certain grossencharacters of E; we then obtain companion forms on the Shimura curves before twisting back again.

In order to apply the ideas of [CV92] it is necessary to have semistable models of the Shimura curves we use, and we construct these following [KM85]. The arguments in [CV92] depend crucially on the use of q-expansions, which are not available to us due to the lack of cusps on our Shimura curves. It has therefore been necessary to construct arguments that apply more generally; we do this by systematic use of expansions at supersingular points.

In section 2 we construct the semistable models of Shimura curves that we need. We work here in rather greater generality than we require for the rest of the paper, assuming nothing about the ramification of p in F. Section 3 contains the necessary background material on automorphic forms

and base change. In section 4 we construct Hecke operators on Shimura curves and present our generalisations of the arguments of [CV92]. Finally, we present the proof of Theorem A in section 5.

Where possible, our notation follows that of the original papers [Car86a] and [CV92]. One possible point of confusion is that the unitary groups we study are, loosely speaking, forms of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2 \times \mathrm{GL}_1$. When we refer to "the" Galois representation attached to such an automorphic form, we mean the Galois representation attached to the GL_2-part, rather than some twist of this by the character corresponding to the GL_1-part. However, we will always ensure that the GL_1-part of our forms is trivial at $\mathfrak{p}$, so the local representation at $\mathfrak{p}$ will in any case be independent of any such twist.

This paper is almost entirely the author's PhD thesis under the supervision of Kevin Buzzard, and it is a pleasure to thank him both for suggesting the problem and for numerous helpful conversations. The proof of the combinatorial result needed in Theorem 4.9 is based on an argument of Noam Elkies; any deficiencies in its exposition are due to me. It is a pleasure to thank Fred Diamond, Frazer Jarvis, and Richard Taylor for several helpful conversations over the past two years.

2 Shimura Curves

2.1 Notation for Shimura Curves

Our notation follows that of [Car86a]. Let F be a totally real field of degree $d > 1$ over $\mathbf{Q}$, and denote by $\tau_1, \ldots, \tau_d$ the infinite places of F. Let $\mathfrak{p}$ be a finite place of F, and let κ be the residue field of F at $\mathfrak{p}$, with cardinality q and characteristic p, and write $\mathcal{O}_\mathfrak{p}$ for the ring of integers of $F_\mathfrak{p}$, the completion of F at $\mathfrak{p}$. Write also $\mathcal{O}_{(\mathfrak{p})}$ for $F \cap \mathcal{O}_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}^{nr}$ for the completion of the ring of integers of $F_\mathfrak{p}^{nr}$, the maximal unramified extension of $F_\mathfrak{p}$. Fix once and for all an isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$; this isomorphism will be used implicitly in our discussion of automorphic forms. Let the primes of F above p be $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$, where $\mathfrak{p}_1 = \mathfrak{p}$. Let B be a quaternion algebra over F which splits at exactly one infinite place, say τ_1, and suppose that B splits at $\mathfrak{p}$. We fix a maximal order $\mathcal{O}_B$ of B, and choose an isomorphism between $\mathcal{O}_{B,v}$ and $\mathcal{M}_2(\mathcal{O}_v)$ at all finite places v of F where B splits.

Define $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$, a reductive group over $\mathbf{Q}$. Then if K is a compact open subgroup of $G(\mathbf{A}_\mathbf{Q}^\infty)$ we define the associated *Shimura curve* to be $M_K(\mathbf{C}) = G(\mathbf{Q}) \backslash \left(G(\mathbf{A}_\mathbf{Q}^\infty) \times (\mathbf{C} - \mathbf{R}) \right) / K$. By work of Shimura $M_K(\mathbf{C})$ has a canonical model M_K over F (see page 152 of [Car86a]). Let Γ be the restricted direct product of the $(B \otimes F_v)^\times$ for all $v \neq \mathfrak{p}$. If $K = K_\mathfrak{p} K^\mathfrak{p}$, with $K_\mathfrak{p} \subseteq \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ and $K^\mathfrak{p} \subseteq \Gamma$, we write $M_{K_\mathfrak{p}, K^\mathfrak{p}}$ for M_K, and we extend this notation in an obvious fashion in the below. Define the groups

$$W_n(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} \mathfrak{p}^n) \right\},$$

and write $M_{n,H}$ for $M_{W_n(\mathfrak{p}),H}$. We also write $M_{0,H}$ for $M_{\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}),H}$. Carayol ([Car86a]) shows further that for H sufficiently small (depending on i) there are smooth models $\mathbf{M}_{i,H}$ for the $M_{i,H}(\mathbf{C})$ over $\mathcal{O}_{\mathfrak{p}}$.

Define the group

$$bal.U_1(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \middle| a - 1 \in \mathfrak{p}, d - 1 \in \mathfrak{p}, c \in \mathfrak{p} \right\}.$$

We construct integral models for $\mathbf{M}_{bal.U_1(\mathfrak{p}),H}$, via constructing integral models for the associated unitary curves. It is these unitary curves that will be used in the main part of this paper.

Carayol also defines $T = \mathrm{Res}_{F/\mathbf{Q}}(\mathbf{G}_m)$, and for any compact open subgroup $U \subseteq T(\mathbf{A}_{\mathbf{Q}}^{\infty})$ the finite set $\mathcal{M}_U(\mathbf{C}) = T(\mathbf{Q}) \backslash \left(T(\mathbf{A}_{\mathbf{Q}}^{\infty}) \times \pi_0(T(\mathbf{R})) \right) / U$. This has a natural right action of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$, acting through $\mathrm{Gal}(F^{ab}/F)$ via the inverse of the natural action of $\pi_0(T(\mathbf{Q}) \backslash T(\mathbf{A}_{\mathbf{Q}}))$ and the isomorphism given by local class field theory (normalised to take geometric Frobenius to a uniformiser), which gives rise to a finite F-scheme together with an action of $T(\mathbf{A}_{\mathbf{Q}}^{\infty})$. Similarly to the above, we define $U_{\mathfrak{p}}^n$ to be the subgroup of $\mathcal{O}_{\mathfrak{p}}^{\times}$ consisting of units congruent to 1 modulo $\mathfrak{p}^n$, and for $V \subseteq (\mathbf{A}_F^{\infty,\mathfrak{p}})^*$ an open compact subgroup we put $\mathcal{M}_{n,V} = \mathcal{M}_{U_{\mathfrak{p}}^n \times V}$, and define $\mathcal{M}_n = \varprojlim_V \mathcal{M}_{n,V}$. Then, for example, $\mathcal{M}_0$ is isomorphic to $\mathrm{Spec}(F_{\mathfrak{p}}^{nr})$.

2.2 Carayol's Work

The curves M_K are not PEL Shimura curves, so in order to construct integral models Carayol instead works with the Shimura curves associated to certain unitary groups, and then uses results of Deligne to relate these to the original curves.

Choose $\lambda < 0$ in $\mathbf{Q}$ so that $K = \mathbf{Q}(\sqrt{\lambda})$ is split at p, and define $E = F(\sqrt{\lambda})$. Fix a choice of square root of λ in $\mathbf{C}$, so that the embeddings $\tau_i : F \hookrightarrow \mathbf{R}$ extend to embeddings $\tau_i : E \hookrightarrow \mathbf{C}$; we always consider E as a subfield of $\mathbf{C}$ via τ_1. Choose a square root μ of λ in $\mathbf{Q}_p$, so that the morphism $E \longrightarrow F_p \oplus F_p$, $x + y\sqrt{\lambda} \mapsto (x + y\mu, x - y\mu)$ extends to an isomorphism

$$E \otimes \mathbf{Q}_p \xrightarrow{\sim} F_p \oplus F_p \xrightarrow{\sim} (F_{\mathfrak{p}_1} \oplus \cdots \oplus F_{\mathfrak{p}_m}) \oplus (F_{\mathfrak{p}_1} \oplus \cdots \oplus F_{\mathfrak{p}_m})$$

which gives an inclusion of E in $F_{\mathfrak{p}}$ via

$$E \hookrightarrow E \otimes \mathbf{Q}_p \xrightarrow{\sim} F_p \otimes F_p \xrightarrow{pr_1} F_p \xrightarrow{pr_1} F_{\mathfrak{p}}.$$

Let $z \longrightarrow \bar{z}$ denote conjugation in E with respect to F. Put $D = B \otimes_F E$ and let $l \longrightarrow \bar{l}$ be the product of the canonical involution of B with conjugation in E. Choose $\delta \in D$ such that $\overline{\delta} = \delta$ and define an involution on D by $l^* = \delta^{-1}\bar{l}\delta$. Choose $\alpha \in E$ such that $\overline{\alpha} = -\alpha$. Then if V denotes the underlying $\mathbf{Q}$-vector space of D with left action of D, we have a symplectic form Ψ on V given by

$$\Psi(v, w) = \mathrm{tr}_{E/\mathbf{Q}}(\alpha \, \mathrm{tr}_{D/E}(v\delta w^*)).$$

Then Ψ is a nondegenerate alternating form on V satisfying, for all $l \in D$,

$$\Psi(lv, w) = \Psi(v, l^* w).$$

Let G' be the reductive algebraic group over $\mathbf{Q}$ such that for any $\mathbf{Q}$-algebra R, $G'(R)$ is the group of D-linear symplectic similitudes of $(V \otimes_{\mathbf{Q}} R, \Psi \otimes_{\mathbf{Q}} R)$; alternatively, we may define G' as follows, as in section 2.1 of [Car86a]. Let $T_E = \mathrm{Res}_{E/\mathbf{Q}}(\mathbf{G}_m)$. Let $\mathbf{S}$ denote $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)$, and let U_E be the subgroup of T_E defined by the equation $z\overline{z} = 1$. Then we can define $G'' = G \times_Z T_E$, and a morphism

$$G'' = G \times_Z T_E \xrightarrow{\nu'} T \times U_E$$

by $(g, z) \mapsto (\nu(g)z\overline{z}, z/\overline{z})$. Then if T' is the subtorus $\mathbf{G}_m \times U_E$ of $T \times U_E$, G' is the inverse image under ν' of T'. This description has the advantage of furnishing us with an isomorphism

$$G'(\mathbf{Q}_p) = \mathbf{Q}_p^* \times \mathrm{GL}_2(F_{\mathfrak{p}}) \times (B \otimes_F F_{\mathfrak{p}_2})^* \times \cdots \times (B \otimes_F F_{\mathfrak{p}_m})^*,$$

where the $\mathbf{Q}_p^*$ factor is given by $\nu(g)z\overline{z}$.

Carayol ([Car86a], section 2.1) defines a morphism $h' : \mathbf{S} \longrightarrow G'_{\mathbf{R}}$, such that the $G'(\mathbf{R})$-conjugacy class of h' may be identified with the complex upper half plane, and the composition $\mathbf{S} \xrightarrow{h'} G'_{\mathbf{R}} \longrightarrow \mathrm{GL}(V_{\mathbf{R}})$ defines a Hodge structure of type $\{(-1, 0), (0, -1)\}$ on $V_{\mathbf{R}}$. We can (and do) choose δ so that Ψ is a polarisation for this Hodge structure.

Let $\mathcal{O}_D$ be a maximal order of D, corresponding to a lattice $V_{\mathbf{Z}}$ in V. The above decomposition of $E \otimes \mathbf{Q}_p$ induces decompositions of $D \otimes \mathbf{Q}_p$ and $\mathcal{O}_D \otimes \mathbf{Z}_p$:

$$\begin{array}{ccccccccc}
\mathcal{O}_D \otimes \mathbf{Z}_p & = & \mathcal{O}_{D_1^1} & \oplus \cdots \oplus & \mathcal{O}_{D_m^1} & \oplus & \mathcal{O}_{D_1^2} & \oplus \cdots \oplus & \mathcal{O}_{D_m^2} \\
\bigcap & & \bigcap & & \bigcap & & \bigcap & & \bigcap \\
D \otimes \mathbf{Q}_p & = & D_1^1 & \oplus \cdots \oplus & D_m^1 & \oplus & D_1^2 & \oplus \cdots \oplus & D_m^2
\end{array}$$

where each D_j^k is an $F_{\mathfrak{p}_j}$-algebra isomorphic to $B \otimes_F F_{\mathfrak{p}_j}$, and $l \mapsto l^*$ interchanges D_j^1 and D_j^2. In particular D_1^1 and D_1^2 are isomorphic to $\mathrm{M}_2(F_p)$. We can, and do, choose $\mathcal{O}_D$, α and δ so that the following conditions hold:

1. $\mathcal{O}_D$ is stable under $l \mapsto l^*$.

2. Each $\mathcal{O}_{D_j^k}$ is a maximal order in D_j^k, and $\mathcal{O}_{D_1^2} \hookrightarrow D_1^2 = \mathrm{M}_2(F_\mathfrak{p})$ identifies with $\mathrm{M}_2(\mathcal{O}_\mathfrak{p})$.

3. Ψ takes integer values on $V_{\mathbf{Z}}$.

4. Ψ induces a perfect pairing Ψ_p on $V_{\mathbf{Z}_p} = V_{\mathbf{Z}} \otimes \mathbf{Z}_p$.

Then every $\mathcal{O}_D \otimes \mathbf{Z}_p$-module Λ admits a decomposition as

$$\Lambda = \Lambda_1^1 \oplus \cdots \oplus \Lambda_m^1 \oplus \Lambda_1^2 \oplus \cdots \oplus \Lambda_m^2$$

with Λ_j^k an $\mathcal{O}_{D_j^k}$-module. The $\mathcal{O}_{D_1^2}$-module Λ_1^2 decomposes further as the direct sum of two $\mathcal{O}_p$-modules $\Lambda_1^{2,1}$ and $\Lambda_1^{2,2}$, the kernels of the idempotents $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ respectively.

Let X' be the $G'(\mathbf{R})$-conjugacy class of h'; then for any open compact subgroup $K' \subseteq G'(\mathbf{A}_{\mathbf{Q}}^{\infty})$ we have a Shimura curve over $\mathbf{C}$

$$M'_{K'}(\mathbf{C}) = G'(\mathbf{Q}) \backslash \left(G'(\mathbf{A}_{\mathbf{Q}}^{\infty}) \times X' \right) / K'.$$

By work of Shimura this has a canonical smooth and proper model $M'_{K'}$ defined over E, which represents the functor

$$\mathcal{M}_{K'} : \underline{E\text{-algebras}} \longrightarrow \underline{\text{Sets}}$$

where for any E-algebra R, $\mathcal{M}_{K'}(R)$ is the set of isomorphism classes of quadruples $(A, \iota, \lambda, \overline{\eta})$, such that

1. A is an abelian scheme of relative dimension $4d$ over R, with an action $\iota : D \hookrightarrow \mathrm{End}(A) \otimes \mathbf{Q}$ of D. This action induces an action of E on $\mathrm{Lie}(A)$, and for each τ_i we let $\mathrm{Lie}_{\tau_i}(A) = \mathrm{Lie}(A) \otimes_{E,\tau_i} \mathbf{C}$. Then we require that $\mathrm{Lie}_{\tau_i} = 0$ for all $i > 1$.

2. λ is a polarisation of A so that the Rosati involution sends $\iota(l)$ to $\iota(l^*)$.

3. $\overline{\eta}$ is a class modulo K' of symplectic D-linear similitudes $\eta : \hat{V}(A) \longrightarrow V \otimes \mathbf{A}_{\mathbf{Q}}^{\infty}$, where $\hat{V} = \hat{T} \otimes \mathbf{Q}$ is the product of the Tate modules of A over all primes, with symplectic structure coming from the Weil pairings.

Our choice of embedding of E into $F_\mathfrak{p}$ allows us to base change this model to $F_\mathfrak{p}$, where we again denote it by $M'_{K'}$. This again represents a moduli problem, and in fact Carayol constructs an integral model for $M'_{K'}$ by describing a moduli problem over $\mathcal{O}_\mathfrak{p}$ which is represented by a smooth and proper scheme $\mathbf{M}'_{K'}$, assuming that K' is small enough, and which satisfies $\mathbf{M}'_{K'} \otimes F_\mathfrak{p} \xrightarrow{\sim} M'_{K'}$. Using the above notation we now describe the moduli problem represented by $\mathbf{M}'_{n,H'}$ for $H' \subseteq \Gamma'$ sufficiently small, where

$$G'(\mathbf{A}_{\mathbf{Q}}^{\infty}) = \mathbf{Q}_p^* \times \mathrm{GL}_2(F_\mathfrak{p}) \times \Gamma',$$

so that $\Gamma' = G'(\mathbf{A}_{\mathbf{Q}}^{p,\infty}) \times (B \otimes_F F_{\mathfrak{p}_2})^* \times \cdots \times (B \otimes_F F_{\mathfrak{p}_m})^*$. In fact, $\mathbf{M}'_{n,H'}$ represents the functor

$$\boldsymbol{\mathcal{M}}'_{n,H'} : \underline{\mathcal{O}_{\mathfrak{p}}\text{-algebras}} \longrightarrow \underline{\text{Sets}}$$

where for any $\mathcal{O}_{\mathfrak{p}}$-algebra R, $\boldsymbol{\mathcal{M}}'_{n,H'}(R)$ is the set of isomorphism classes of quintuples $(A, \iota, \lambda, \eta_{\mathfrak{p}}, \overline{\eta}^{\mathfrak{p}})$ such that

1. A is an abelian scheme of relative dimension $4d$ over R, with an action $\iota : \mathcal{O}_D \hookrightarrow \mathrm{End}_R(A)$ of $\mathcal{O}_D$ such that
   (a) the projective R-module $\mathrm{Lie}_1^{2,1}(A)$ has rank one, and $\mathcal{O}_{\mathfrak{p}}$ acts on it via $\mathcal{O}_{\mathfrak{p}} \hookrightarrow R$,
   (b) for $j \geq 2$, $\mathrm{Lie}_j^2(A) = 0$.
2. λ is a polarisation of A of degree prime to p such that the Rosati involution sends $\iota(l)$ to $\iota(l^*)$.
3. $\eta_{\mathfrak{p}}$ is an isomorphism of $(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)$-modules

$$\eta_{\mathfrak{p}} : (A_{\mathfrak{p}^n})_1^{2,1} \xrightarrow{\sim} (\mathfrak{p}^{-n}/\mathcal{O}_{\mathfrak{p}})^2.$$

4. $\overline{\eta}^{\mathfrak{p}}$ is a class of isomorphisms $\eta^{\mathfrak{p}} = \eta_p^{\mathfrak{p}} \oplus \eta^p : T_p^{\mathfrak{p}}(A) \oplus \hat{T}^p(A) \xrightarrow{\sim} W_p^{\mathfrak{p}} \oplus \hat{W}^{\mathfrak{p}}$ modulo H', with $\eta_p^{\mathfrak{p}}$ linear and η^p symplectic, where $\hat{T}^p(A)$ is the product of the Tate modules away from p, $T_p^{\mathfrak{p}}(A) = (T_p(A))_2^2 \oplus \cdots \oplus (T_p(A))_m^2$, $\hat{W}^p = V_{\mathbf{Z}} \otimes \hat{\mathbf{Z}}^p$ and $W_p^{\mathfrak{p}} = (V_{\mathbf{Z}_p})_2^2 \oplus \cdots \oplus (V_{\mathbf{Z}_p})_m^2$.

We have a short exact sequence

$$1 \longrightarrow G_1 \longrightarrow G' \xrightarrow{\nu'} T' \longrightarrow 1$$

where G_1 is the derived subgroup of G, and thus also of G'. Then for any compact open subgroup $U' \subseteq T(\mathbf{A}_{\mathbf{Q}}^{\infty})$ we define the finite set

$$\mathcal{M}'_{U'}(\mathbf{C}) = T'(\mathbf{Q}) \backslash \left(T'(\mathbf{A}_{\mathbf{Q}}^{\infty}) \times \pi_0(T'(\mathbf{R})) \right) / U'.$$

This has a natural right action of $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$ defined via class field theory as before, and we have a model over E, denoted $\mathcal{M}'_{U'}$. For $K' \subseteq G'(\mathbf{A}_{\mathbf{Q}}^{\infty})$ open and compact this yields an E-morphism with geometrically connected fibres

$$M'_{K'} \xrightarrow{\nu'} \mathcal{M}'_{\nu'(K')}.$$

For H' sufficiently small so that $M'_{0,H'}$ exists, we see that there is a universal abelian variety A together with an action of $\mathcal{O}_D$. Then for all n we have a locally free $(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)$-scheme $E'_{n,H'}$ defined by

$$E'_{n,H'} = (A_{\mathfrak{p}^n})_1^{2,1}.$$

In section 3.3 of [Car86a] Carayol notes the equality

$$E'_{n,H'} = [M'_{n,H'} \times (\mathfrak{p}^{-n}/\mathcal{O}_\mathfrak{p})^2]/\operatorname{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n);$$

we can also define a group L'_n over $\mathcal{M}'_{0,V'}$ via

$$L'_n = [\mathcal{M}'_{n,V'} \times (\mathfrak{p}^{-n}/\mathcal{O}_\mathfrak{p})]/(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)^*.$$

Locally for the étale topology on $M'_{0,H'}$ we see that $E'_{n,H'}$ is isomorphic to the constant $(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$-module $(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)^2$, and $M'_{n,H'}$ is the scheme over $M'_{0,H'}$ parameterising isomorphisms $k_\mathfrak{p} : E'_{n,H'} \cong (\mathfrak{p}^{-n}/\mathcal{O}_\mathfrak{p})^2$. Upon choosing a uniformiser $\mathfrak{p}$ we have an isomorphism

$$\bigwedge_{\mathcal{O}_\mathfrak{p}}^{2} (\mathfrak{p}^{-n}/\mathcal{O}_\mathfrak{p})^2 \xrightarrow[\sim]{\det} (\mathfrak{p}^{-2n}/\mathfrak{p}^{-n}) \xrightarrow[\sim]{\mathfrak{p}^n} (\mathfrak{p}^{-n}/\mathcal{O}_\mathfrak{p})$$

which induces an isomorphism

$$\bigwedge_{\mathcal{O}_\mathfrak{p}}^{2} E'_n \xrightarrow{\sim} \nu'^* L'_n.$$

We denote the resulting alternating $\mathcal{O}_\mathfrak{p}$-bilinear pairing (the "Weil pairing") by

$$e'_n : E'_n \times E'_n \longrightarrow \nu'^* L'_n.$$

These definitions are easily extended to $\mathbf{M}'_{0,H'}$. Indeed, if $\mathbf{A}$ is the universal abelian scheme over $\mathbf{M}'_{0,H'}$, we put

$$\mathbf{E}'_{n,H'} = (\mathbf{A}_{\mathfrak{p}^n})_1^{2,1}.$$

These fit together to give a divisible $\mathcal{O}_\mathfrak{p}$-module $\mathbf{E}'_\infty$. Let $\boldsymbol{\mathcal{M}}'_{n,V'}$ be the normalisation of $\mathcal{M}'_{n,V'}$ in the ring of $\mathfrak{p}$-integers of E. Then ([Car86a], section 8.3) there is a unique simultaneous extension of the groups $L'_{n,V'}$ to finite locally free groups $\mathbf{L}'_{n,V'}$ over $\boldsymbol{\mathcal{M}}'_{n,V'}$ satisfying obvious compatablities, and a unique extension of $M'_{K'} \xrightarrow{\nu'} \mathcal{M}'_{\nu'(K')}$ to a morphism $\mathbf{M}'_{K'} \xrightarrow{\boldsymbol{\nu}'} \boldsymbol{\mathcal{M}}'_{\nu'(K')}$. In section 9.1 of [Car86a] Carayol demonstrates that there is a unique extension of e'_n to an alternating $\mathcal{O}_\mathfrak{p}$-bilinear pairing

$$\mathbf{e}'_n : \mathbf{E}'_n \times_{\mathbf{M}'_{0,H'}} \mathbf{E}'_n \longrightarrow \boldsymbol{\nu}'^* \mathbf{L}'_n.$$

At each geometric point x of $\mathbf{M}'_{0,H'} \otimes \kappa$ we have a local divisible $\mathcal{O}_\mathfrak{p}$-module $\mathbf{E}'_\infty|_x$. For every positive integer h there is a unique divisible formal $\mathcal{O}_\mathfrak{p}$-module Σ_h over $\bar{\kappa}$ of height h (see [Car86a], section 0.8). From the existence of an alternating pairing we conclude that $\mathbf{E}'_\infty|_x$ must be self-dual, and up to isomorphism either

(1) $\mathbf{E}'_\infty|_x \cong \Sigma_1 \times (F_\mathfrak{p}/\mathcal{O}_\mathfrak{p})$, or
(2) $\mathbf{E}'_\infty|_x \cong \Sigma_2$.

As in the modular setting, we call x *ordinary* in the first case and *supersingular* in the second. From section 9.4 of [Car86a] it follows that the set of supersingular points is finite and nonempty.

For the purposes of our constructions (and in particular the modular interpretation of Hecke operators) it is more natural to work with the unitary curves; we will make use of results from [Jar99], which deals instead with the quaternionic curves, but it is easy to see that the arguments in question carry over unchanged from the quaternion algebra to the unitary setting.

2.3 The $bal.U_1(\mathfrak{p})$-Problem

Definition 2.1. Let S be a scheme over $\mathbf{M}'_{0,H'}$, where H' is sufficiently small for $\mathbf{E}'_{1,H'}$ to exist. For a subscheme Q of $\mathbf{E}'_1|_S$, let $\mathcal{I}_Q$ be the ideal sheaf defining Q. If P is a section of $\mathbf{E}'_1|_S$ and $\mathcal{K}$ a locally free sub-$\mathcal{O}_\mathfrak{p}/\mathfrak{p}$-group scheme of $\mathbf{E}'_1|_S$, we say that P *generates* $\mathcal{K}$, and write $\mathcal{K} = \langle P\rangle$, if $\mathcal{K}$ is the subscheme of $\mathbf{E}'_1|_S$ defined by the ideal $\prod_{\lambda\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p})}\mathcal{I}_{\lambda P}$.

Note that a generator in our sense is an "$(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$-generator" in the sense of [KM85].

Definition 2.2. A $bal.U_1(\mathfrak{p})$*-structure* on S, an $\mathbf{M}'_{0,H'}$-scheme, is an f.p.p.f. short exact sequence of $\mathcal{O}_\mathfrak{p}$-group schemes on S

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathbf{E}'_1\big|_S \longrightarrow \mathcal{K}' \longrightarrow 0$$

such that $\mathcal{K}$, $\mathcal{K}'$ are both locally free of rank q, together with $P \in \mathcal{K}(S)$, $P' \in \mathcal{K}'(S)$ such that $\mathcal{K} = \langle P\rangle$, $\mathcal{K}' = \langle P'\rangle$.

Definition 2.3. Define the functor

$$\begin{aligned}\boldsymbol{\mathcal{M}}'_{bal.U_1(\mathfrak{p}),H'} : \underline{\mathrm{Sch}/\mathbf{M}'_{0,H'}} &\longrightarrow \underline{\mathrm{Sets}},\\ T &\mapsto bal.U_1(\mathfrak{p})(T) := \{bal.U_1(\mathfrak{p})\text{-structures on T}\}.\end{aligned}$$

Definition 2.4. Define the functor

$$\mathcal{M}'_{bal.U_1(\mathfrak{p}),H'} : \underline{\mathrm{Sch}/M'{}_{0,H'}} \longrightarrow \underline{\mathrm{Sets}},$$

$T \mapsto \{P$ a nowhere-zero section of $E'_1\big|_T$, P' a nowhere-zero section of $E'_1\big|_T/\langle P\rangle\}$.

Lemma 2.5. *The functors $\boldsymbol{\mathcal{M}'}_{bal.U_1(\mathfrak{p}),H'}$ and $\mathcal{M}'{}_{bal.U_1(\mathfrak{p}),H'}$ agree on $M'_{0,H'}$-schemes.*

Proof. To give an $\boldsymbol{\mathcal{M}}'_{bal.U_1(\mathfrak{p}),H'}$-structure is to give a section P which generates a finite flat sub-$\mathcal{O}_\mathfrak{p}$-group scheme $\langle P\rangle$ of $\mathbf{E}'_1|_S$ of rank q, together with a generating section P' of $\mathbf{E}'_1|_S/\langle P\rangle$. If S is a scheme over $M'_{0,H'}$,

$\mathbf{E}'_1|_S = E'_1|_S$ which is étale over S. Thus locally in the étale topology $\mathbf{E}'_1|_S$ is non-canonically isomorphic to $(\mathcal{O}_\mathfrak{p}/\mathfrak{p})^2$, so that $\langle P\rangle$, $\langle P'\rangle$ are both isomorphic to $(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$, and thus P, P' are both nowhere zero sections killed by $\mathfrak{p}$. The converse is clear. □

Lemma 2.6. *The F-scheme $M'_{bal.U_1(\mathfrak{p}),H'}$ represents the functor $\mathcal{M}'_{bal.U_1(\mathfrak{p}),H'}$.*

Proof. Let $\mathcal{M}'_{1,H'}$ be the functor:

$$\mathcal{M}'_{1,H'} : \underline{\mathrm{Sch}/M'_{0,H'}} \longrightarrow \underline{\mathrm{Sets}},$$

$$T \mapsto \{\text{pairs } (P,Q) \text{ of sections of } \left.E'_1\right|_T \text{ over } T \text{ which trivialise } \left.E'_1\right|_T\}.$$

Then $\mathcal{M}'_{1,H'}$ is represented by $M'_{1,H'}$. For any object S of $(Sch/M'_{0,H'})$ we have an action of $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ on $\mathcal{M}'_{1,H'}(S)$ given by

$$(P,Q) \mapsto (P,Q)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aP+cQ, bP+dQ).$$

Furthermore this action is clearly functorial, and the equivalence classes under the action of the subgroup

$$bal.\widetilde{U}_1(\mathfrak{p}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) \middle| a-1\in\mathfrak{p}, d-1\in\mathfrak{p}, c\in\mathfrak{p}\right\}$$

give elements of $\mathcal{M}'_{bal.U_1(\mathfrak{p}),H'}(S)$ in the obvious way, so that we have a map of moduli problems $\mathcal{M}'_{1,H'} \longrightarrow \mathcal{M}'_{bal.U_1(\mathfrak{p}),H'}$.

Locally for the étale topology we may complete any $bal.U_1(\mathfrak{p})$-structure (P,P') (note that the pair (P,P') determines $\mathcal{K}$, $\mathcal{K}'$) to a pair $(P,Q) \in \mathcal{M}_{1,H}(S)$ trivialising $E'_1|_S$, and so to a morphism $S \longrightarrow M'_{1,H'}$ of $M'_{0,H'}$-schemes. Thus to give a $bal.U_1(\mathfrak{p})$-structure on S is to give a section $S \longrightarrow M'_{1,H'}/bal.\widetilde{U}_1(\mathfrak{p}) = M'_{bal.U_1(\mathfrak{p}),H'}$, and $\mathcal{M}'_{bal.U_1(\mathfrak{p}),H'}$ is represented by the F-scheme $M'_{bal.U_1(\mathfrak{p}),H'}$. □

Lemma 2.7. *The functor $\boldsymbol{\mathcal{M}}'_{bal.U_1(\mathfrak{p}),H'}$ is represented by the scheme $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$.*

Proof. By Lemma 7.5 of [Jar99] the functor $\boldsymbol{\mathcal{M}}'_{U_1(\mathfrak{p}),H'}$ is representable, so it suffices to prove that the moduli problem associating $bal.U_1(\mathfrak{p})$-structures to $U_1(\mathfrak{p})$-structures is relatively representable. The additional structure is the choice of a generator for $\mathbf{E}'_1|_S/\langle P\rangle$, so the result follows by Proposition 1.10.13 of [KM85]. □

Lemma 2.8. *The scheme $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$ is regular of dimension two, and the projection map $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'} \longrightarrow \mathbf{M}'_{0,H'}$ is finite and flat.*

Proof. (cf. [KM85] 5.1, [Jar99] Theorem 7.6). That the natural projection map $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'} \to \mathbf{M}'_{0,H'}$ is finite is immediate from the finiteness of $\mathbf{E}'_1$, and the observation that any point P is a generator for only finitely many $\mathcal{K}$. To see this, note that it suffices to prove this over an algebraically closed base field; but then the result follows from the proof of [Dri76], Proposition 1.7, which gives an explicit description of $\mathbf{E}'_1$, and from an argument similar to the one on page 241 of [DR73]. We prove flatness and regularity via the homogeneity principle of [KM85]. Let U be the set of points x of $\mathbf{M}'_{0,H'}$ such that for any lift $y \in \mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$ of x the local ring at y is regular and flat over the local ring at x. We prove that:

(H1) U is open.

(H2) $M_{bal.U_1(\mathfrak{p}),H'}$ is finite étale over $M_{0,H'}$, so in particular (as $M_{0,H'}$ is regular of dimension two) U contains all of $M_{0,H'}$.

(H3) If U contains an ordinary point of $\mathbf{M}'_{0,H'} \otimes \kappa$ then it contains all ordinary points of $\mathbf{M}'_{0,H'} \otimes \kappa$.

(H4) If U contains a supersingular point of $\mathbf{M}'_{0,H'} \otimes \kappa$ then it contains all supersingular points of $\mathbf{M}'_{0,H'} \otimes \kappa$.

(H5) U contains a supersingular point of $\mathbf{M}'_{0,H'} \otimes \kappa$.

It will then be immediate that $U = \mathbf{M}'_{0,H'}$ (because U is open and there are only finitely many supersingular points, (H5) implies that U also contains an ordinary point, and hence all points by (H3) and (H4)), so the lemma will follow (two-dimensionality being clear, as $\mathbf{M}'_{0,H'}$ is itself regular two-dimensional)).

The projection $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'} \longrightarrow \mathbf{M}'_{0,H'}$ is finite so proper, so the complement of U is closed, being the union of the two closed sets in $\mathbf{M}'_{0,H'}$ which are the images of the closed subsets of $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$ at which $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$ is not regular or not flat over $\mathbf{M}'_{0,H'}$. Thus (H1) is proved.

Since $M'_{1,H'}$ is certainly finite étale over $M'_{0,H'}$, so too is $M'_{bal.U_1(\mathfrak{p}),H'}$, so (H2) is immediate.

The proofs of (H3) and (H4) are identical to those in [Jar99] Theorem 7.6, but we include them for completeness' sake. Let x be a closed point of $\mathbf{M}'_{0,H'} \otimes \kappa$. If y is a closed point of $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$ lying above x, then the map $\mathcal{O}_{\mathbf{M}'_{0,H'},x} \longrightarrow \mathcal{O}_{\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'},y}$ is flat if and only if the induced map $\widehat{\mathcal{O}^{sh}}_{\mathbf{M}'_{0,H'},x} \longrightarrow \widehat{\mathcal{O}^{sh}}_{\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'},y}$ is flat, and $\mathcal{O}_{\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'},y}$ is regular if and only if $\widehat{\mathcal{O}^{sh}}_{\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'},y}$ is regular ([Gro67], IV 18.8.8 and 18.8.13). We are thus reduced, by the argument on p.133 of [KM85], to considering the case where x is a geometric point of the special fibre.

Let $(\widehat{\mathbf{M}'}_{0,H'})_{(x)}$ be the completion of the strict henselisation of $\mathbf{M}'_{0,H'}$ at x. Then by [Car86a] 6.6 $\mathbf{E}'_\infty|_{(\widehat{\mathbf{M}'}_{0,H'})_{(x)}}$ is the universal deformation of $\mathbf{E}'_\infty|_x$, so that the isomorphism class of the map $\widehat{\mathcal{O}^{sh}}_{\mathbf{M}'_{0,H'},x} \longrightarrow \widehat{\mathcal{O}^{sh}}_{\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'},y}$ depends only on the universal deformation of $\mathbf{E}'_\infty|_x$, which in turn depends

only on whether x is ordinary or supersingular, as required.

It remains to prove (H5). Accordingly, let x be a supersingular point of the special fibre of $\mathbf{M}'_{0,H'}$. Let $\mathcal{C}$ denote the category of complete noetherian local $\mathcal{O}_{\mathfrak{p}}^{nr}$-algebras with residue field $\bar{\kappa}$. By [Jar99] Theorem 4.5, the functor

$$\begin{aligned} &\mathcal{C} \longrightarrow \underline{\text{Sets}}, \\ &R \mapsto \{\text{isomorphism classes of deformations of } \mathbf{E}'_\infty\big|_x\}, \end{aligned}$$

is represented by $\mathcal{O}_{\mathfrak{p}}^{nr}[[t]]$.

Lemma 2.9. *The set $bal.U_1(\mathfrak{p})(x)$ contains precisely one element.*

Proof. The set $bal.U_1(\mathfrak{p})(x)$ is the set of triples $(\mathcal{K}, P, P')$ with $\mathcal{K}$ a finite flat $\mathcal{O}_{\mathfrak{p}}$-subgroup scheme of $\mathbf{E}'_1|_{\bar{\kappa}}$, $P \in \mathcal{K}(\bar{\kappa})$ a generator of $\mathcal{K}$, and P' a $\bar{\kappa}$-valued generator of $\mathcal{K}' = \mathbf{E}'_1|_x/\mathcal{K}$. But $\mathbf{E}'_1|_x$ is local (as we are in the supersingular case), so $\mathcal{K}$ and $\mathcal{K}'$ are both local, so $P = P' = 0$. This gives us a unique $bal.U_1(\mathfrak{p})$-structure, as required (again, by the proof of Proposition 1.7 of [Dri76] Σ_2 has a unique sub $(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})$-module of rank q). □

Consequently, we see that the moduli problem of $bal.U_1(\mathfrak{p})$-structures on $\mathbf{E}'_\infty|_{(\widehat{\mathbf{M}'}_{0,H'})_{(x)}}/\mathcal{O}_{\mathfrak{p}}^{nr}[[t]]$ is represented by an affine scheme $\operatorname{Spec}(A)$ with A a local ring. By the argument of [KM85] Proposition 5.2.2 we need only show that the maximal ideal of A is generated by two elements, which we can do by mimicking the proof of [KM85] Theorem 5.3.2. In fact, the required argument is formally identical to that given in [KM85]; again, we can work with parameters $X(P)$, $X'(P')$ on the formal $\mathcal{O}_{\mathfrak{p}}$-module, and then apply Proposition 5.3.4 of [KM85] in the case $p^n = q$. □

2.4 Canonical Balanced Structures

Just as in the modular curve case, the existence of an alternating form on $\mathbf{E}'_1$ allows us to define canonical $bal.U_1(\mathfrak{p})$-structures. Firstly, we define

Definition 2.10. A section P of $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H}|_S$, S an $\mathbf{M}'_{0,H'}$-scheme, is a *generator* of $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}(S)$ if the subscheme of $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}(S)$ defined by the ideal $\prod_{\lambda\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})}\mathcal{I}_{\lambda P}$ is $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}(S)$.

Theorem 2.11. *The functor*

$$\begin{aligned} &\underline{Sch/\mathbf{M}'_{0,H'}} \longrightarrow \underline{Sets}, \\ &S \mapsto \{\textit{generators of } \boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}(S)\} \end{aligned}$$

is representable by an affine scheme $\mathbf{L}'^{\times}_{1,H'}$.

Proof. In fact, $\mathbf{L}'^{\times}_{1,H'}$ is obviously a closed subscheme of the (finite, so) affine scheme $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}$. □

Definition 2.12. Given a $bal.U_1(\mathfrak{p})$-structure $(\mathcal{K}, P, P')$ we define $\langle P, P' \rangle := \mathbf{e}_1(P, Q)$ where Q is any lift of P' to $\mathbf{E}'_1|_S$ (note that this is well defined because $\mathbf{e}_1(P, P) = 1$; note also that Q may in fact be defined over a finite flat extension of S, but $\mathbf{e}_1(P, Q)$ will still be defined over S by descent).

We can now define canonical $bal.U_1(\mathfrak{p})$-structures exactly as in [KM85] chapter 9:

Definition 2.13. A *$bal.U_1(\mathfrak{p})^{can}$-structure* on S, an $\mathbf{L}'^{\times}_{1,H'}$-scheme, is an f.p.p.f. short exact sequence of $\mathcal{O}_p$-group schemes on S

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathbf{E}'_1|_S \longrightarrow \mathcal{K}' \longrightarrow 0$$

such that $\mathcal{K}$, $\mathcal{K}'$ are both locally free of rank q, together with $P \in \mathcal{K}(S)$, $P' \in \mathcal{K}'(S)$ such that the $\mathbf{L}'^{\times}_{1,H'}$-structure on S provided by P is the canonical one (i.e. the one that S possesses as an $\mathbf{L}'^{\times}_{1,H'}$-scheme).

The obvious modifications of the proofs of Proposition 9.1.7 and Corollaries 9.1.8–9.1.10 of [KM85] are also valid in our case. Thus

Theorem 2.14. *The functor*

$$\begin{aligned} \underline{Sch/\mathbf{L}'^{\times}_{1,H'}} &\longrightarrow \underline{Sets}, \\ S &\mapsto \{bal.U_1(\mathfrak{p})^{can} - \textit{structures on } S\} \end{aligned}$$

is represented by a scheme $\mathbf{M}'^{can}_{bal.U_1(\mathfrak{p}),H'}$ regular and equidimensional of dimension two.

Proof. Exactly as in Chapter 9 of [KM85]. □

Note that after a change of base to the tamely ramified extension of $F_\mathfrak{p}$ determined by our choice of uniformiser $\mathfrak{p}$, the scheme $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}$ is étale, corresponding in the usual fashion to $\mathfrak{p}^{-1}/\mathcal{O}_\mathfrak{p}$ together with a Galois action. Indeed, from section 8.2 of [Car86a] we see that if $F^0_\mathfrak{p}$ is the extension of $F^{nr}_\mathfrak{p}$ corresponding to $(\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*$ via class field theory, then $\boldsymbol{\nu}'^*\mathbf{L}'_{1,H'}$ corresponds to $\mathfrak{p}^{-1}/\mathcal{O}_\mathfrak{p}$, together with the action of $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F^{nr}_\mathfrak{p})$ given by the composition

$$\mathrm{Gal}(\overline{F}_\mathfrak{p}/F^{nr}_\mathfrak{p}) \longrightarrow \mathrm{Gal}(F^0_\mathfrak{p}/F^{nr}_\mathfrak{p}) \cong (\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*.$$

2.5 Igusa Curves

Our study of the special fibre of $\mathbf{M}'_{bal.U_1(\mathfrak{p}),H'}$ makes use of Igusa curves; these are defined in just the same way as in the modular case. If S is a $\mathbf{M}'_{0,H'} \otimes \kappa$-scheme, then we have the absolute Frobenius morphism $F_{\text{abs}} : S \longrightarrow S$, given by $s \mapsto s^q$ on affine rings. We also in the usual way obtain for any S-scheme Z a relative Frobenius $F : Z \longrightarrow Z^{(\sigma)}$, where $Z^{(\sigma)}$ is the pullback of Z via $F_{\text{abs}} : S \longrightarrow S$. In particular, we have a map $F : \mathbf{E}'_1|_S \longrightarrow \mathbf{E}'_1|_S^{(\sigma)}$.

Let $(-)^D$ denote Cartier duality. Then we define the *Verschiebung* $V : \mathbf{E}'_1|_S^{(\sigma)} \longrightarrow \mathbf{E}'_1|_S$ to be the dual of

$$F : (\mathbf{E}'_1|_S)^D \longrightarrow (\mathbf{E}'_1|_S^D)^{(\sigma)} = (\mathbf{E}'_1|_S^{(\sigma)})^D.$$

Definition 2.15. Let S be a $\mathbf{M}'_{0,H'} \otimes \kappa$-scheme. Then an *Igusa structure* on S is a point $P \in \mathbf{E}'_1|_S^{(\sigma)}(S)$ which generates the kernel of $V : \mathbf{E}'_1|_S^{(\sigma)} \longrightarrow \mathbf{E}'_1|_S$.

Lemma 2.16. *The moduli problem*

$$\begin{aligned} \mathrm{Ig} : \underline{Sch/\mathbf{M}'_{0,H'} \otimes \kappa} &\longrightarrow \underline{Sets}, \\ S &\mapsto \{\textit{Igusa structures on } S\} \end{aligned}$$

is representable by a finite flat $\mathbf{M}'_{0,H'} \otimes \kappa$*-scheme* $\mathbf{M}'_{\mathrm{Ig},H'}$*, which is regular one-dimensional, of rank* $q-1$ *over* $\mathbf{M}'_{0,H} \otimes \kappa$*.*

Proof. Everything follows as in the proof of Theorem 6.1.1. of [KM85] (see sections 8 and 9 of [Jar99]), except for regularity, which follows from an obvious modification of the argument on page 363 of [KM85]. ☐

2.6 The Special Fibre

We now analyse the special fibre of $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$ with the aid of the crossing theorem of [KM85], 13.1.3. We will show that $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'} \otimes \kappa$ is the union of two smooth curves crossing transversally above the supersingular points of $\mathbf{M}'_{0,H'} \otimes \kappa$. As in the modular case, one of these curves is essentially a copy of $\mathbf{M}'_{0,H'} \otimes \kappa$, and corresponds to the generators of $\ker F$, and the other is an Igusa curve, the scheme of generators of $\ker V$.

There is a unique morphism $\mathbf{L}'^{\times}_{1,H'} \longrightarrow \kappa$ (this can be seen explicitly by consideration of the construction of tame extensions via adjoining roots of uniformisers) so the reduction mod $\mathfrak{p}$ of the $bal.U_1(\mathfrak{p})^{can}$ problem is given by

$$\begin{aligned} \underline{\mathrm{Sch}/\mathbf{M}'_{0,H'} \otimes \kappa} &\longrightarrow \underline{\mathrm{Sets}} \\ S &\mapsto \{bal.U_1(\mathfrak{p})\text{-structures } (\mathcal{K}, P, P') \text{ with } \langle P, P' \rangle = 1\}. \end{aligned}$$

The condition that the pairing be 1 is a closed condition, so this problem is representable, say by $\mathbf{M}'_{bal.U_1(\mathfrak{p});det=1,H'}$.

We need to analyse the possible $(bal.U_1(\mathfrak{p}); det = 1)$-structures on geometric points of $\mathbf{M}'_{0,H'} \otimes \kappa$. From section 9 of [Jar99] we see that if x is supersingular the only possibility for $\mathcal{K}$ is $\ker F$, and if x is ordinary then $\mathcal{K}$ is either $\ker F$ or $\ker V$. Furthermore:

Lemma 2.17. *Let S be a $\mathbf{M}'_{0,H'} \otimes \kappa$-scheme. Let P be a generator of* $\ker V$*. Then the triple* $(\ker V, P, 0)$ *is a* $(bal.U_1(\mathfrak{p}); det = 1)$*-structure on* $S^{(\sigma)}$ *and the triple* $(\ker F, 0, P \bmod \ker F)$ *is a* $(bal.U_1(\mathfrak{p}); det = 1)$*-structure on S. Furthermore, these constructions define closed immersions* $\mathbf{M}'_{\mathrm{Ig},H'} \hookrightarrow \mathbf{M}'^{(\sigma)}_{bal.U_1(\mathfrak{p});det=1,H'}$ *and* $\mathbf{M}'_{\mathrm{Ig},H'} \hookrightarrow \mathbf{M}'_{bal.U_1(\mathfrak{p});det=1,H'}$ *respectively.*

Proof. That these define $bal.U_1(\mathfrak{p})$-structures on S follows from the proof of [Jar99] Theorem 10.2. We need to check that $\langle P, 0\rangle = 1$; but by definition $\langle P, 0\rangle = \mathbf{e}_1(P, P) = 1$, as required.

Our claimed immersions are certainly $\mathbf{M}'_{0,H'} \otimes \kappa$-maps between finite $\mathbf{M}'_{0,H'} \otimes \kappa$-schemes, so are finite and hence proper. We claim that they are injective on S-valued points for all $\mathbf{M}'_{0,H'} \otimes \kappa$-schemes; this is immediate, since we can clearly recover P from the image. But ([Gro67] 18.12.6) a proper monomorphism is a closed immersion, as required. $\square$

Theorem 2.18. $\mathbf{M}'_{bal.U_1(\mathfrak{p});det=1,H'}$ *is the disjoint union with transverse crossings at the supersingular points of two smooth κ-curves.*

Proof. This follows from an application of the Crossings Theorem ([KM85], Theorem 13.1.3) and the previous lemma; one of our curves is $\mathbf{M}'_{\mathrm{Ig},H'}$ and the other is $\mathbf{M}'^{(\sigma^{-1})}_{\mathrm{Ig},H'}$. The required argument is very similar to that in section 9 of [Jar99]. That the crossings occur at the supersingular points is simply the statement that $\ker F = \ker V$ precisely in the supersingular case.

We claim that the completion of the strict henselisation of the local ring at the supersingular points is $\bar{\kappa}[[x, y]]/(xy)$, where as before $x = X(P)$, $y = X'(P')$, from which it is immediate that the crossings at the singular points are transverse.

From the crossings theorem it suffices to show that our two components are given by $x = 0$ and $y = 0$ respectively. But one component is given by $P = 0$, which occurs if and only if $x = X(P) = 0$, and similarly the other component is defined by $y = 0$, as required. $\square$

In fact, we claim that the universal formal deformation at a supersingular point is given by $F^0_{\mathfrak{p}}[[x, y]]/(xy - \pi)$, where π is a uniformiser for $F^0_{\mathfrak{p}}$. The proof of this again follows exactly as in the proof of theorem 5.3.2 of [KM85], which shows that the universal formal deformation is given by $F^0_{\mathfrak{p}}[[x, y]]/(f)$ for some f, and that the maximal ideal of $F^0_{\mathfrak{p}}[[x, y]]$ is generated by x, y and f; the additional information that the strict henselisation of the local ring mod p is $\bar{\kappa}[[x, y]]/(xy)$ gives the required result.

3 Automorphic Forms

3.1 Definitions

We say that π is a Hilbert modular form of weight $k \geq 2$ if for each $\tau : F \hookrightarrow \mathbf{R}$ the representation π_τ is the $(k-1)$-st lowest discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with central character $a \mapsto a^{2-k}$ (note that we are only working with parallel weight modular forms). We will also say that π_∞ is of weight k when this condition holds.

Fix an isomorphism $\overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$, and let "reduction modulo p" mean reduction modulo the maximal ideal of $\mathcal{O}_{\overline{\mathbf{Q}}_p}$.

Definition 3.1. A *mod p Hilbert modular form* is an equivalence class of Hilbert modular forms, where two forms π, π' are equivalent if they have the same mod p Galois representations (see below); equivalently, if for all finite places $v \nmid p$ of F at which π_v and π'_v are unramified principal series, say $\pi_v = \pi(\psi_1, \psi_2)$, $\pi'_v = \pi(\psi'_1, \psi'_2)$, we have an equality

$$\{\psi_1(\mathrm{Frob}_v), \psi_2(\mathrm{Frob}_v)\} = \{\psi'_1(\mathrm{Frob}_v), \psi'_2(\mathrm{Frob}_v)\} \bmod p.$$

Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$, and let $U_1(\mathfrak{n})$ denote the subgroup of $\prod \mathrm{GL}_2(\mathcal{O}_{F,v})$ consisting of elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \in \mathfrak{n}$ and $(a-1) \in \mathfrak{n}$. Then (by the theory of the conductor) for any Hilbert modular form π there exists some $\mathfrak{n}$ for which $\pi^{U_1(\mathfrak{n})} \neq 0$, and we say that π has level $\mathfrak{n}$. Note that a modular form has infinitely many levels, all divisible by the minimal level, the product of the local conductors of π. We say that a mod p form has weight k and level $\mathfrak{n}$ if some form in the equivalence class defining it does.

We have, as usual, a notion of a Hilbert modular form being ordinary at $\mathfrak{p}_i$; if $\pi_{\mathfrak{p}_i}$ is unramified with Satake parameters α, β, then we let $a_{\mathfrak{p}_i} = q^{1/2}(\alpha + \beta)$. Then $a_{\mathfrak{p}_i}$ is the eigenvalue at $\mathfrak{p}_i$ of the classical Hilbert modular form corresponding to π, and we say that π is ordinary if $a_{\mathfrak{p}_i} \neq 0 \bmod p$ for all $\mathfrak{p}_i | p$. There is an obvious notion of an ordinary mod p form; we simply demand that it is the reduction mod p of a form which is ordinary at $\mathfrak{p}$. Then:

Lemma 3.2. *An ordinary mod p Hilbert modular form of level $\mathfrak{n}$ prime to p is an ordinary mod p form of weight 2 and level $\mathfrak{n}p$.*

Proof. This is an easy application of Hida theory, and in fact follows at once from Theorem 3 of [Wil88]. □

Let π be a Hilbert modular form, and let M be the field of definition of π (that is, the fixed field of the automorphisms σ of $\mathbf{C}$ satisfying $\sigma\pi \cong \pi$). It is known that M is either a totally real or a CM number field, and that for each prime p of $\mathbf{Q}$ and embedding $\lambda : M \hookrightarrow \overline{\mathbf{Q}}_p$ there is a continuous irreducible representation

$$\rho_{\pi,\lambda} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(M_\lambda)$$

determined (thanks to the Cebotarev density theorem and the Brauer-Nesbitt theorem) by the following property: if $v \nmid p$ is a place of F such that π_v is unramified then $\rho_{\pi,\lambda}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is unramified with Frob_v having minimal polynomial

$$X^2 - t_v X + (\mathbf{N}v)s_v$$

where t_v is the eigenvalue of the Hecke operator

$$\left[\mathrm{GL}_2(\mathcal{O}_{F_v})\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{F_v})\right]$$

(where ϖ_v is a uniformiser of $\mathcal{O}_{F_v}$) on $\pi^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ and s_v is the eigenvalue of the operator

$$\left[\mathrm{GL}_2(\mathcal{O}_{F_v})\begin{pmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{F_v})\right].$$

As usual, by the compactness of $\mathrm{Gal}(\overline{F}/F)$ we may conjugate $\rho_{\pi,\lambda}$ to a representation valued in $\mathrm{GL}_2(\mathcal{O}_{M,\lambda})$, and then reduce modulo the maximal ideal of $\mathcal{O}_{M,\lambda}$ to get a continuous representation to $GL_2(\overline{\mathbf{F}}_p)$. The semisimplification of this representation is well-defined, and we denote it by $\overline{\rho}_{\pi,p}$.

In particular, the above discussion shows that there is a continuous mod p Galois representation $\overline{\rho}_\pi$ canonically associated to any mod p Hilbert modular form π, determined as above by the characteristic polynomials of Frobenius.

Let π be a weight 2 level $\mathfrak{n}p$ Hilbert modular form, corresponding to our initial weight k level $\mathfrak{n}$ mod p Hilbert modular form. In order to apply the level lowering results below, we need to assume that it is not the case that $[F(\zeta_p):F]=2$ and $\overline{\rho}_\pi|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ is reducible; in the case that this does not hold it is easy to construct the companion form "by hand", so from now on we ignore this case until we treat it in the proof of Theorem A. At various places in our arguments on Shimura curves we will need to assume that the level is sufficiently large (equivalently, the compact open subgroup of $G'(\mathbf{A}_{\mathbf{Q}}^\infty)$ corresponding to the level structure is sufficiently small). This may be accomplished via a trick originally due to Diamond and Taylor; namely, we choose a prime $\mathfrak{q} \nmid \mathfrak{n}p$ such that there are no congruences between forms of level $U_1(\mathfrak{n})$ and $\mathfrak{q}$-new forms of level dividing $U_1(\mathfrak{n}) \cap U_1^1(\mathfrak{q})$, where $U_1^1(\mathfrak{q})$ is the subgroup of $\prod \mathrm{GL}_2(\mathcal{O}_{F,v})$ consisting of elements $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c, a-1, d-1 \in \mathfrak{q}$. We then work throughout with an auxillary $U_1^1(\mathfrak{q})$-level structure, which we can remove at the end due to the lack of congruences. There are infinitely many such primes $\mathfrak{q}$; see the remark following Lemma 12.2 of [Jar99] for a proof of this.

If d is even, we assume that there is a finite place $z \nmid p$ of F such that π_z is not principal series. Fortunately, this does not entail a loss of generality. By Theorem 1 of [Tay89] there is a finite place $z \nmid \mathfrak{n}p$ where $\overline{\rho}_\pi$ is unramified, $\mathbf{N}_{F/\mathbf{Q}}(z) \equiv -1 \bmod p$, and a Hilbert modular form $\tilde{\pi}$ of the same weight as

π and level $U_1(\mathfrak{n}) \cap U_z$ (where U_z is the subgroup of $\prod \mathrm{GL}_2(\mathcal{O}_{F,v})$ consisting of elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $z|c$) such that $\tilde{\pi}_z$ is unramified special and $\overline{\rho}_{\tilde{\pi}} \cong \overline{\rho}_{\pi}$. We can then work throughout with $\tilde{\pi}$ in place of π, and remove the auxillary level z structure at the end thanks to Theorem A of [Fuj99].

We now show how to construct a holomorphic differential corresponding to π on a unitary Shimura curve. We choose a quadratic imaginary extension $K/\mathbf{Q}$ as in section 2.2, but in addition to requiring it to split p, we require that it splits all primes $l|\mathfrak{n}$, and that it is disjoint from the extension of F given by the kernel of the Galois representation $\overline{\rho}_{\pi}$ (this last condition ensures that the base change of π to E is cuspidal; see Theorem 3.4 below). We also require that all ramified places of F and B split in E.

Definition 3.3. If π is an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_F)$, and Π is an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_E)$, then Π is a *base change lift* of π, denoted $BC_{E/F}(\pi)$, if for every place v of F, and every $w|v$, the Langlands parameter attached to Π_w equals the restriction to W_{E_w} of the Langlands parameter $\sigma_v : W_{F_v} \to \mathrm{GL}_2(\mathbf{C})$ of π_v. In particular, if v splits in E then $\Pi_w \cong \pi_v$.

Theorem 3.4. 1. *Every cuspidal representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ has a unique base change lift to $\mathrm{GL}_2(\mathbf{A}_E)$; the lift is itself cuspidal unless π is monomial of the form $\pi\left(\mathrm{Ind}_{W_E}^{W_F} \theta\right)$.*

2. *If π, π' have the same base change lift to $\mathrm{GL}_2(\mathbf{A}_E)$ then $\pi' \cong \pi \otimes \omega$ for some character ω of $F^\times N_{E/F}(\mathbf{A}_E^\times)\backslash \mathbf{A}_F^\times$.*

3. *A cuspidal representation Π of $\mathrm{GL}_2(\mathbf{A}_E)$ equals $BC_{E/F}(\pi)$ for some π if and only if Π is invariant under the action of $\mathrm{Gal}(E/F)$.*

Proof. [Lan80]. $\square$

Let $\pi_E = BC_{E/F}(\pi)$. Let B denote the indefinite quaternion algebra over F which if d is odd is ramified only at $\tau_2, \ldots, \tau_d$, and which if d is even is ramified precisely at $\tau_2, \ldots, \tau_d$ and some $z \nmid \mathfrak{n}p$ with π_z not principal series.

We now, as in [BR89], twist π_E by a character η so that $\eta \otimes (\eta \circ c) = \chi_\pi^{-1} \circ N_{E/F}$, where c denotes the nontrivial element of $\mathrm{Gal}(E/F)$ (the existence of such characters is easily deduced from the arguments in the proof of Lemma VI.2.10 of [HT01]). Then if $^\vee$ denotes the contragredient representation we have

$$\begin{aligned}
(\pi_E \otimes \eta)^\vee \circ c &\cong (\pi_E \circ c)^\vee \otimes (\eta \circ c)^\vee \\
&\cong \pi_E^\vee \otimes (\eta \circ c)^\vee \\
&\cong (\pi_E \otimes \chi_{\pi_E}^{-1}) \otimes (\eta \circ c)^{-1} \\
&\cong \pi_E \otimes (\chi_\pi^{-1} \circ N_{E/F}) \otimes (\eta \circ c)^{-1} \\
&\cong \pi_E \otimes \eta.
\end{aligned}$$

Now, $(\chi_\pi)_\infty = 1$, so we may suppose that η is trivial at the archimedean places. Furthermore, we choose η so that $\eta_{\mathfrak{p}_i}$ is trivial at all $\mathfrak{p}_i|p$, where we identify each $\mathfrak{p}_i$ with the place of E given by our fixed choice of a place of K dividing p.

Then by the Jacquet-Langlands theorem (Theorem 3.5) there is a unique irreducible automorphic representation Π of $(D\otimes\mathbf{A_Q})^\times$ (where $D = B\otimes E$) such that $JL(\Pi) = \pi_E\otimes\eta$. Furthermore (cf. page 199 of [HT01]) the condition that $(\pi_E\otimes\eta)^\vee\circ c \cong \pi_E\otimes\eta$ gives $\Pi^*\cong\Pi$.

Now from Theorem VI.2.9 and Lemma VI.2.10 of [HT01] we see that there is a character ψ of $\mathbf{A}_K^\times/K^\times$ and an automorphic representation π_1 of $G'(\mathbf{A_Q})$ such that $BC(\pi_1) = (\psi,\Pi)$ and $(\psi)_\infty$ is trivial. Furthermore, from Theorem 3.6 below we see that $(\pi_1)^{K'}\neq 0$, where K' is defined as follows: $K'_l = G'(\mathbf{Z}_l)$ for all $l\nmid\mathfrak{n}p$. For $l|\mathfrak{n}$ we have an isomorphism $G'(\mathbf{Q}_l) = \mathbf{Q}_l^*\times\prod_{\mathfrak{l}|l}\mathrm{GL}_2(F_\mathfrak{l})$, and we put $K'_l = \mathbf{Z}_l\times\prod_{\mathfrak{l}|l}\left\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathcal{O}_\mathfrak{l}) : c, d-1\equiv 0 \bmod \mathfrak{l}^{v_\mathfrak{l}(\mathfrak{n})}\right\}$. Finally, $K'_p = \mathbf{Z}_p\times\prod_{\mathfrak{p}_i|p} bal.U_1(\mathfrak{p}_i)$. We will sometimes denote the away-from-$\mathfrak{p}$ part of the level by H'.

3.2 Base Change

We now recall various results on automorphic forms due to, amongst others, Jacquet, Langlands, Clozel and Kottwitz. A convenient reference for these results is [HT01], where they are all stated in far greater generality; the reader should note, however, that many of the results quoted have simpler proofs in our special cases than those in [HT01].

Theorem 3.5. *(Jacquet-Langlands) If ρ is an irreducible automorphic representation of $(D\otimes\mathbf{A_Q})^\times$ then there is a unique automorphic representation $JL(\rho)$ of $GL_2(\mathbf{A}_E)$ which occurs in the discrete spectrum and for which $JL(\rho)^{S(D)}\cong\rho^{S(D)}$, where $S(D)$ is the set of places at which D ramifies. The image of JL is the set of irreducible automorphic representations π of $GL_2(\mathbf{A}_E)$ which are special or supercuspidal at all places in $S(D)$.*

Proof. [JL70]. □

Let π be an irreducible automorphic representation of $G'(\mathbf{A_Q})$, and let x be a place of $\mathbf{Q}$ which splits in K. Then as in section 2.2 we have an isomorphism $\mathbf{Q}_x \xrightarrow{\sim} E_y$ as an E-algebra, and an identification $G'(\mathbf{Q}_x)\cong B_y^\times\times\mathbf{Q}_x^\times$. Accordingly, we have a decomposition $\pi_x\xrightarrow{\sim}\pi_y\otimes\psi_{\pi,y^c}$, where c denotes complex conjugation. Let $BC(\pi_x)$ be the representation

$$\pi_y\otimes\pi_{y^c}\otimes(\psi_{\pi,y^c}\circ c)\otimes(\psi_{\pi,y}\circ c)$$

of

$$\begin{aligned} G'(E_x) &\cong B_x^\times\times E_x^\times \\ &\cong B_y^\times\times B_{y^c}^\times\times E_y^\times\times E_{y^c}^\times. \end{aligned}$$

There is also, in [HT01] a definition of $BC(\pi_x)$ for all but finitely many places of $\mathbf{Q}$ which are inert in E. We will not need this definition, except to note that by our choice of E there is in fact a definition for *all* inert places, and that the base change of an unramified representation is unramified (this follows easily from the discussion on page 199 of [HT01]).

Theorem 3.6. *Suppose that π is an irreducible automorphic representation of $G'(\mathbf{A}_{\mathbf{Q}})$ such that π_∞ has weight 2. Then there is a unique irreducible automorphic representation $BC(\pi) = (\psi, \Pi)$ of $\mathbf{A}_K^\times \times (D \otimes \mathbf{A}_{\mathbf{Q}})^\times$ such that*

1. $\psi = \psi_\pi|^c_{\mathbf{A}_K^\times}$.
2. *If x is a place of $\mathbf{Q}$ then $BC(\pi)_x = BC(\pi_x)$.*
3. Π_∞ *has weight 2.*
4. $\psi_\Pi|_{\mathbf{A}_K^\times} = \psi^c/\psi$.
5. $\Pi^* \cong \Pi$, *where* $\Pi^*(g) = \Pi(g^{-*})$.

Proof. Immediate from Theorem VI.2.1 of [HT01]. □

3.3 Galois representations

We recall a version of Matsushima's formula (see page 8 of [HT01] and page 420 of [Car86b]). For $i = 0, 1, 2$ we have

$$\mathrm{H}^i_{\acute{e}t}(M'_{K'} \times \overline{E}, \overline{\mathbf{Q}}_p) = \oplus_\pi \pi^{K'} \otimes R^i(\pi),$$

where the sum is over irreducible admissible representations π of $G'(\mathbf{A}_{\mathbf{Q}})$ with π_∞ of weight 2, and $R^i(\pi)$ is a certain finite dimensional continuous representation of $Gal(\overline{E}/E)$.

Define a virtual representation

$$[R(\pi)] = \sum_{i=0}^{2} (-1)^{i+1} [R^i(\pi)].$$

It can be shown that

$$[R(\pi)] \neq 0$$

(see the remarks after Theorem 1 in [Kot92]). In the cases of interest to us, $R^0(\pi) = R^2(\pi) = 0$; in fact we have

Theorem 3.7. *Let π be an irreducible admissible representation of $G'(\mathbf{A}_{\mathbf{Q}})$ over $\overline{\mathbf{Q}}_p$ with π_∞ of weight 2, and suppose that $BC(\pi) = (\psi, \Pi)$ with $JL(\Pi)$ cuspidal. Then $R^0(\pi) = R^2(\pi) = 0$.*

Proof. This is a special case of Corollary VI.2.7 of [HT01]. □

Let m_π denote the multiplicity of π in the space of automorphic forms on $G'(\mathbf{Q})\backslash G'(\mathbf{A_Q})$ transforming by ψ_π under the centre of $G'(\mathbf{A_Q})$. Then

Lemma 3.8. $\dim[R(\pi)] = 2m_\pi = 2$.

Proof. The first equality follows at once from Theorem 1.3.1 of [Har00] and the remarks after Theorem 1 in [Kot92]. The statement that $m_\pi = 1$ may be found on page 132 of [CL99]. $\square$

Thus under the assumption that $BC(\pi) = (\psi, \Pi)$ with $JL(\Pi)$ cuspidal, we have attached a continuous 2-dimensional p-adic representation $R^1(\pi)$ to π (note that we have implicitly used the theory of the conductor for $U(1,1)$ developed in section 5 of [LR04]).

By continuity $R^1(\pi)$ is defined over a finite extension K_π of $\mathbf{Q}_p$, and (from the compactness of $\mathrm{Gal}(\overline{E}/E)$) there is a $\mathrm{Gal}(\overline{E}/E)$-stable lattice L in $R^1(\pi)$. Denote the semisimplification of the representation $L/m_{K_\pi}L$ (where m_{K_π} is the maximal ideal of $\mathcal{O}_{K_\pi}$) by $\overline{\rho}_\pi$. It is not *a priori* obvious that $\overline{\rho}_\pi$ is independent of the choice of L, but this follows from Theorem 3.9 below; by the Cebotarev density theorem and the Brauer-Nesbitt theorem (recalling that $p > 2$) $\overline{\rho}_\pi$ is determined by the knowledge of $\mathrm{tr}(\overline{\rho}_\pi(\mathrm{Frob}_y))$ for all but finitely many y lying above primes which split in K.

Before stating Theorem 3.9 it is convenient to recall the connection between holomorphic differentials on $M'_{K'}$ and the first étale cohomology of $M'_{K'}$. We have the Hodge-theoretic decomposition

$$\begin{aligned}\mathrm{H}^0(M'_{K'} \otimes \mathbf{C}, \Omega^1_{M'_{K'}}) \oplus \mathrm{H}^0(M'_{K'} \otimes \mathbf{C}, \overline{\Omega}^1_{M'_{K'}}) &\xrightarrow{\sim} \mathrm{H}^1(M'_{K'} \otimes \mathbf{C}, \mathbf{C}) \\ &\xrightarrow{\sim} \mathrm{H}^1_{\acute{e}t}(M'_{K'} \times \overline{E}, \overline{\mathbf{Q}}_p),\end{aligned}$$

from which a standard argument shows that an automophic form π as above corresponds to a unique differential ω_π on $M'_{K'}$, which is an eigenvector for the Hecke operators $T_\mathfrak{l}$. Specifically, this correspondence is determined by the requirement that at all places $\mathfrak{l}$ of E at which $\pi_\mathfrak{l}$ is unramified principal series and at which we have defined $T_\mathfrak{l}$, the eigenvalue of $T_\mathfrak{l}$ acting on ω_π is $\mathbf{N}\mathfrak{l}^{-1/2}$ multiplied by the sum of the Satake parameters of $\pi_\mathfrak{l}$ (see for example §1.5 of [HT02]). Then we have

Theorem 3.9. *Suppose that $K' = U_1(\mathfrak{n})$, $y \nmid \mathfrak{n}p$ is a prime of E lying over a place of $\mathbf{Q}$ which splits in K, and π_y is unramified principal series. If $T_y\omega_\pi = a_y\omega_\pi$ and $S_y\omega_\pi = b_y\omega_\pi$ then*

$$\mathrm{tr}(\overline{\rho}_\pi(\mathrm{Frob}_y)) = \overline{a_y/\psi_\pi(\mathrm{Frob}_y)}$$

and

$$\det(\overline{\rho}_\pi(\mathrm{Frob}_y)) = \overline{b_y/\psi_\pi(\mathrm{Frob}_y)},$$

where ψ_π is the central character of π.

Proof. This is immediate from Corollary VII.1.10 of [HT01]. $\square$

Corollary 3.10. *If ω_f is a differential on $\mathbf{M}'^{can}_{bal.U_1(\mathfrak{p}),H'}$ which is an eigenvector for the operators $T_{\mathfrak{l}}$ with eigenvalues $a_{\mathfrak{l}}$, then there is a continuous representation*

$$\overline{\rho}_f : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$$

such that for all but finitely many $\mathfrak{l}$ lying over primes of $\mathbf{Q}$ which split in K we have

$$\mathrm{tr}(\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{l}})) = a_{\mathfrak{l}}.$$

Proof. By the usual Deligne-Serre minimal prime lemma (Lemma 6.11 of [DS74]) there is a differential ω_π whose Hecke eigenvalues lift those of ω_f; then the required representation is obtained by twisting the definition of $\overline{\rho}_\pi$ by $\overline{\psi}_\pi$. $\square$

4 Construction of Companion Forms

4.1 Hecke operators

The results of this section are valid for the general case where $\mathfrak{p}$ unramified. We define Hecke operators on $M'_{K'}$ in the familiar way (namely as double cosets). We then give a modular interpretation of these operators, which allows us to extend their definitions to the integral models. Indeed, let $\mathfrak{l}$ be a prime of F lying over the prime $l \neq p$ of $\mathbf{Q}$, and suppose that B is split at $\mathfrak{l}$. Suppose also that l splits in $\mathbf{Q}(\sqrt{\lambda})$. Then if $\mathfrak{l}_1 = \mathfrak{l}, \mathfrak{l}_2, \dots, \mathfrak{l}_k$ are the primes of F above l, we have $G'(\mathbf{Q}_l) \xrightarrow{\sim} \mathbf{Q}_l^* \times \mathrm{GL}_2(F_{\mathfrak{l}}) \times \mathrm{GL}_2(F_{\mathfrak{l}_2}) \times \cdots \times \mathrm{GL}_2(F_{\mathfrak{l}_k})$. Suppose now that K' is of the form $\prod_q K'_q$, and that $K'_l = \mathbf{Z}_l \times \prod_{i=1}^k \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{l}_i})$. Let $\varpi_{\mathfrak{l}}$ be an element of $\mathbf{A}^\infty_{\mathbf{Q}}$ which is a uniformiser at $\mathfrak{l}$ and 1 everywhere else; then we define

$$T_{\mathfrak{l}} = \left[K' \begin{pmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & 1 \end{pmatrix} K'\right].$$

In order to compute the action of $T_{\mathfrak{l}}$ we express it as the ratio of two other Hecke operators $X_{\mathfrak{l}}$ and $Y_{\mathfrak{l}}$; define $X_{\mathfrak{l}} = [K'x_{\mathfrak{l}}K']$ and $Y_{\mathfrak{l}} = [K'y_{\mathfrak{l}}K']$ where with obvious notation we have

$$x_{\mathfrak{l}} = \left(l^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{l}}^{-1} \end{pmatrix}, \begin{pmatrix} \varpi_{\mathfrak{l}_2}^{-1} & 0 \\ 0 & \varpi_{\mathfrak{l}_2}^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \varpi_{\mathfrak{l}_k}^{-1} & 0 \\ 0 & \varpi_{\mathfrak{l}_k}^{-1} \end{pmatrix}\right)$$

and

$$y_{\mathfrak{l}} = \left(l^{-1}, \begin{pmatrix} \varpi_{\mathfrak{l}}^{-1} & 0 \\ 0 & \varpi_{\mathfrak{l}}^{-1} \end{pmatrix}, \begin{pmatrix} \varpi_{\mathfrak{l}_2}^{-1} & 0 \\ 0 & \varpi_{\mathfrak{l}_2}^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \varpi_{\mathfrak{l}_k}^{-1} & 0 \\ 0 & \varpi_{\mathfrak{l}_k}^{-1} \end{pmatrix}\right)$$

and we have $T_{\mathfrak{l}} = X_{\mathfrak{l}} Y_{\mathfrak{l}}^{-1}$.

In the usual way we have induced correspondences $X_{\mathfrak{l}}$ and $Y_{\mathfrak{l}}$ on $\mathbf{M}_{K'}$. In section 7.5 of [Car86a] Carayol computes the action of $G'(\mathbf{A}_{\mathbf{Q}}^{\infty})$ on the inverse limit $\mathbf{M}' = \lim_{\leftarrow K'} \mathbf{M}'_{K'}$, which allows us to compute $X_{\mathfrak{l}}$ and $Y_{\mathfrak{l}}$ on $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$. In the usual way we have an induced action on $\Omega^1_{\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}/\mathcal{O}_{\mathfrak{p}}}$, and Carayol's results allow us to explicate this as

$$\omega | X_{\mathfrak{l}}(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P,Q) \mapsto \sum_{\phi} \omega\left((\phi A, \phi\iota, \phi\lambda, \phi\eta_{\mathfrak{p}}, \phi\overline{\eta}_{\mathfrak{p}}, \phi P, \phi Q)\right),$$

where the sum is over a certain set of $\mathbf{N}\mathfrak{l}+1$ isogenies of degree l^d (we do not need to know the precise details of these isogenies). Similarly, the action of $Y_{\mathfrak{l}}$ is via a single isogeny of degree l^d.

4.2 Igusa curves

We now extend the Hecke action to an action on meromorphic functions on Igusa curves, and relate this action to the action on differentials on Shimura curves defined above. The required notation was introduced in Chapter 2.

Let $\pi : \mathbf{A}' \to \mathbf{M}'_{0,H'}$ be the universal abelian variety. The $\mathcal{O}_{\mathbf{M}'_{0,H'}}$-module $\pi_*\Omega^1_{\mathbf{A}'/\mathbf{M}'_{0,H'}}$ is an $\mathcal{O}_D \otimes \mathbf{Z}_p$-module, and we put

$$\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} = \left(\pi_*\Omega^1_{\mathbf{A}'/\mathbf{M}'_{0,H'}}\right)_1^{2,1}.$$

Since $\mathrm{Lie}_1^{2,1}(\mathbf{A}'_{0,H'})$ is locally free of rank one we see that $\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}}$ is locally free of rank one. Similarly (or by pullback from $\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}}$ using the universal property of $\pi : \mathbf{A}' \to \mathbf{M}'_{0,H'}$) we have a locally free sheaf of rank one $\omega_{(\mathbf{A}')^\vee/\mathbf{M}'_{0,H'}}$. Then we have

Theorem 4.1. *The Kodaira-Spencer map gives a canonical isomorphism*

$$\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} \otimes \omega_{(\mathbf{A}')^\vee/\mathbf{M}'_{0,H'}} \xrightarrow{\sim} \Omega^1_{\mathbf{M}'_{0,H'}/\mathcal{O}_{\mathfrak{p}}}.$$

Proof. This is Proposition 4.1 of [Kas04]. $\square$

We have a morphism $\pi_{Ig} : \mathbf{M}'_{Ig,H'} \to \mathbf{M}'_{0,H'} \otimes \kappa$ of degree $(p-1)$ which is étale over the ordinary locus and totally ramified over the supersingular locus (the former follows from a computation of the local rings, and the latter from the uniqueness of Igusa structures at supersingular points). Let ss denote the divisor of supersingular points on $\mathbf{M}'_{Ig,H'}$, and let s be the degree of this divisor.

Proposition 4.2. $2s = (p-1)\deg \Omega^1_{\mathbf{M}'_{0,H}/\kappa}$.

Proof. By flatness we have $\deg \Omega^1_{\mathbf{M}'_{0,H}/\kappa} = 2(g-1)$, where g is the genus of any geometric fibre of $\mathbf{M}'_{0,H}/\mathcal{O}_{\mathfrak{p}}$, so it suffices to prove that $s = (p-1)(g-1)$. We do this by computing the genus g_0 of ${\mathbf{M}'}_{U_0(\mathfrak{p}),H'}$ in two ways. In characteristic zero the Riemann-Hurwitz formula gives $g_0 = (p+1)(g-1)+1$, whereas on the special fibre we have $g_0 = 2g + s - 1$, from the description of the special fibre of $\mathbf{M}'_{U_0(\mathfrak{p}),H'}$ in section 10 of [Jar99] (or rather from the obvious modification of the argument of [Jar99] to our case). Equating these expressions gives the result. ☐

Put $\omega^+ := \pi^*_{Ig}(\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} \otimes \kappa)$ and $\omega^- := \pi^*_{Ig}(\omega_{(\mathbf{A}')^\vee/\mathbf{M}'_{0,H'}} \otimes \kappa)$.

Proposition 4.3. *ω^+ and ω^- have degree s, and there is a natural isomorphism $\Omega^1_{\mathbf{M}'_{Ig,H'}} \cong \omega^+ \otimes \omega^-((p-2)(ss))$.*

Proof. The degree of the polarisation associated to $\mathbf{A}'$ is prime to p, so it is étale, and thus induces a (non-canonical) isomorphism $\omega^+ \xrightarrow{\sim} \omega^-$, so $\deg \omega^+ = \deg \omega^-$. Then

$$\begin{aligned} 2s &= (p-1) \deg \Omega^1_{\mathbf{M}'_{0,H'}/\kappa} \\ &= 2(p-1) \deg(\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} \otimes \kappa) \end{aligned}$$

so that

$$\begin{aligned} \deg \omega_{\mathbf{M}'_{Ig,H'}} &= \deg(\pi_{Ig}) \deg(\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} \otimes \kappa) \\ &= (p-1) \deg(\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} \otimes \kappa) \\ &= s. \end{aligned}$$

Then by the Riemann-Hurwitz formula we have

$$\begin{aligned} \Omega^1_{\mathbf{M}'_{Ig,H'}} &\cong \pi^*_{Ig} \Omega^1_{\mathbf{M}'_{0,H'}}((p-2)ss) \\ &\cong \pi^*_{Ig}((\omega_{\mathbf{A}'/\mathbf{M}'_{0,H'}} \otimes \kappa) \otimes (\omega_{(\mathbf{A}')^\vee/\mathbf{M}'_{0,H'}} \otimes \kappa))((p-2)ss) \\ &\cong \omega^+ \otimes \omega^-((p-2)ss). \end{aligned}$$

☐

We now define sections a^+ of ω^+ and a^- of ω^-; $a^+ \otimes a^-$ will play the role of a "Hasse invariant". Given the data of an abelian variety with polarisation and level structure $(\pi : A \to S, \iota, \lambda, \eta_{\mathfrak{p}}, \overline{\eta}_{\mathfrak{p}})$, where S is an $\mathbf{F}_p$-scheme, together with an Igusa structure $P \in \ker(V|_{A^\sigma})$, we need to construct an element of $\mathrm{H}^0(A, \Omega^1_{A/S})$.

By Cartier duality, P gives a map $\phi_P : \ker(F|_A) \to \mathbf{G}_m$. The standard invariant differential dX/X on $\mathbf{G}_m$ pulls back to an invariant differential $\phi_P^*(dX/X)$ on $\ker(F|_A)$. Because S is an $\mathbf{F}_p$-scheme, the restriction map

$$(\text{invariant 1-forms on } A/S) \longrightarrow (\text{invariant 1-forms on } \ker(F|_A))$$

is an isomorphism, so there is a unique invariant differential on A whose restriction to $\ker(F|_A)$ is $\phi_P^*(dX/X)$.

Thus we have an element of $\mathrm{H}^0(A, \Omega^1_{A/S}) = \mathrm{H}^0(S, \pi_*\Omega^1_{A/S})$, so an element of $\mathrm{H}^0(S, (\pi_*\Omega^1_{A/S})_1^{2,1})$. Applying this with $S = \mathbf{M}'_{Ig,H'}$ and $A = \mathbf{A}'/S$ (the universal abelian variety defined above) gives the required section a^+ of ω^+. The section a^- of ω^- is defined in the same way.

Proposition 4.4. *a^+ has simple zeroes at the supersingular points, and is nonzero on the ordinary locus.*

Proof. If $A/\overline{\kappa}$ is supersingular, then the only Igusa structure $P \in \ker(F|_A)$ is $P = 0$, so $\phi_P = 0$ and we see that a^+ vanishes at every supersingular point. On the other hand if $A/\overline{\kappa}$ is ordinary then we obviously obtain a nonzero element of $\mathrm{H}^0(S, (\pi_*\Omega^1_{A/S})_1^{2,1})$, so a^+ cannot be identically zero. Since the degree of ω^+ is s, the number of supersingular points, a^+ can only have simple zeroes at the supersingular points, and is nonzero elsewhere. □

We define a Hecke action on $\omega^+ \otimes \omega^-$ by demanding that it commutes with the Kodaira-Spencer isomorphism and the action on differentials defined above.

Proposition 4.5. *If $\nu \otimes \nu'$ is a local section $\omega^+ \otimes \omega^-$, then*

$$((\nu \otimes \nu')X_{\mathfrak{l}})(A, \iota, \lambda, \eta_{\mathfrak{p}}, \overline{\eta}_{\mathfrak{p}}, P) = \frac{1}{l^d} \sum_{\phi} \phi^*\nu|_{\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)} \otimes \phi^*\nu'|_{\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)}$$

where the sum is over the same isogenies used in the explication of $X_{\mathfrak{l}}$ above, and a similar result holds for $Y_{\mathfrak{l}}$.

Proof. We use the following description of the Kodaira-Spencer map: for A/S an abelian scheme, there is a short exact sequence

$$0 \to \Omega^1_{A/S} \to \mathrm{H}^1_{dR}(A/S) \to (\Omega^1_{A^\vee/S})^\vee \to 0,$$

and a cup-product pairing

$$\langle , \rangle_{dR} : \mathrm{H}^1_{dR}(A/S) \times \mathrm{H}^1_{dR}(A/S) \to \mathcal{O}_S.$$

If S is a Σ-scheme for some scheme Σ, we also have the Gauss-Manin connection

$$\nabla : \mathrm{H}^1_{dR}(A/S) \to \mathrm{H}^1_{dR}(A/S) \otimes \Omega^1_{S/\Sigma}$$

(see section III.9 of [FC90]). This gives a map

$$\begin{aligned} \omega^+ \otimes \omega^- &\to \Omega^1_{S/\Sigma} \\ (v \otimes w) &\mapsto \langle v, \nabla w \rangle. \end{aligned}$$

Thus under the Kodaira-Spencer isomorphism the image of

$$\phi^*\nu|_{\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)} \otimes \phi^*\nu'|_{\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)}$$

is the differential

$$\frac{1}{l^d}\sum_{\phi}\langle\phi^*\nu|_{\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)}, \nabla\phi^*\nu'|_{\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)}\rangle.$$

Now, by the naturality of the Gauss-Manin connection we see that if $\phi : A \to A'$ is an isogeny and v' is an invariant differential on A', then $\nabla_A\phi^*v' = \phi^*\nabla_{A'}v'$ in H^1_{dR}. Also, if v is an invariant differential on A, the fact that the adjoint of A with respect to the cup product pairing on de Rham cohomology is $\phi^\vee$, satisfying $\phi^\vee \circ \phi = \deg\phi$, gives

$$\langle\phi^*v, \phi^*\nabla v'\rangle^A_{dR} = \deg\phi\langle v, \nabla v'\rangle^{A'}_{dR}.$$

Every isogeny we sum over has degree l^d, so we obtain (with obvious suppression of additional structures)

$$\begin{aligned}\frac{1}{l^d}\sum_{\phi}\langle\phi^*\nu|_{\phi(A)}, \nabla\phi^*\nu'|_{\phi(A)}\rangle &= \sum_{\phi}\langle\nu|_{\phi(A)}, \nabla\nu'|_{\phi(A)}\rangle \\ &= X_{\mathfrak{l}}(\langle\nu|_{(A)}, \nabla\nu'|_{(A)}\rangle).\end{aligned}$$

□

We now examine two ways of moving between meromorphic functions and meromorphic differentials on the Igusa curve. The first is the map $d : f \mapsto df$, and the second is multiplication by $a^+ \otimes a^-$ followed by application of the Kodaira-Spencer isomorphism.

We define a Hecke action on meromorphic functions on $\mathbf{M}'_{Ig,H'}$ by

$$(X_{\mathfrak{l}}f)(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P) = \frac{1}{l^d}\sum_{\phi}\phi^*f(\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)),$$

and similarly for $Y_{\mathfrak{l}}$.

Theorem 4.6. *$X_{\mathfrak{l}}(df) = l^d d(X_{\mathfrak{l}}f)$, and $Y_{\mathfrak{l}}(df) = l^d d(Y_{\mathfrak{l}}f)$, so that $T_{\mathfrak{l}}(df) = d(T_{\mathfrak{l}}f)$.*

Proof. We have

$$\begin{aligned}l^d d(X_{\mathfrak{l}}f) &= l^d\frac{1}{l^d}d\left(\sum_{\phi}\phi^*f(\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P))\right) \\ &= \sum_{\phi}\phi^*(df)(\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)) \\ &= X_{\mathfrak{l}}(df).\end{aligned}$$

The result for $Y_{\mathfrak{l}}$ follows similarly. □

Theorem 4.7. *$X_{\mathfrak{l}}((a^+ \otimes a^-)f) = (a^+ \otimes a^-)X_{\mathfrak{l}}f$ and $Y_{\mathfrak{l}}((a^+ \otimes a^-)f) = (a^+ \otimes a^-)Y_{\mathfrak{l}}f$, so that $T_{\mathfrak{l}}((a^+ \otimes a^-)f) = (a^+ \otimes a^-)T_{\mathfrak{l}}f$.*

Proof. We compute

$$\begin{aligned} X_{\mathfrak{l}}((a^+ \otimes a^-)f) &= \frac{1}{l^d}\sum_{\phi} \phi^*((a^+ \otimes a^-)f)(\phi(A,\iota,\lambda,\eta_{\mathfrak{p}},\overline{\eta}_{\mathfrak{p}},P)) \\ &= \frac{1}{l^d}\sum_{\phi} \phi^*(a^+)|_{\phi(A,P)} \otimes \phi^*(a^-)|_{\phi(A,P)}\phi^* f|_{\phi(A,P)} \end{aligned}$$

(with obvious suppression of additional structures other than the Igusa structure), and similarly for $Y_{\mathfrak{l}}$. It is thus sufficient to prove that for all isogenies $\phi : (A,P) \to (A',Q)$ we have $\phi^*(a^+|_{(A',Q)}) = a^+|_{(A,P)}$ and $\phi^*(a^-|_{(A',Q)}) = a^-|_{(A,P)}$. We prove the former, the proof of the latter being formally identical. We have maps $\psi_P : \ker(F|_A) \to \mathbf{G}_m$, $\psi_Q : \ker(F|_{A'}) \to \mathbf{G}_m$, and $a^+|_{\ker(F|_A)} = \psi_P^*(dX/X)$, $a^+|_{\ker(F|_{A'})} = \psi_Q^*(dX/X)$. Since $a^+|_A$ is determined by $a^+|_{\ker(F|_A)}$, it suffices to check that $\phi \circ \psi_Q = \psi_P$, which is obvious. □

We now examine an analogue of Atkin-Lehner's U operator. Define, in the same fashion as $T_{\mathfrak{l}}$, an operator

$$U_{\mathfrak{p}} = \left[K' \begin{pmatrix} \varpi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} K'\right]$$

on $\mathbf{M}'_{K'}$, where $\varpi_{\mathfrak{p}}$ is a finite adele which is a uniformiser of $\mathcal{O}_{\mathfrak{p}}$ at $\mathfrak{p}$ and 1 everywhere else. This acts as a correspondence on $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$, and by the functoriality of Néron models we get an induced endomorphism of $\mathrm{Pic}^0(\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'} \otimes \overline{\kappa})$, and in particular of $\mathrm{Pic}^0(\mathbf{M}'_{Ig,H'})$. This action is not, however, given by a correspondence. Let Frob be the Frobenius correspondence on $\mathbf{M}'_{Ig,H'}$, with dual Ver. Then we have

Theorem 4.8. *$U_{\mathfrak{p}} = \mathrm{Ver}$ on $\mathbf{M}'_{Ig,H'}$.*

Proof. The proof is very similar to that of the corresponding result in the classical case, for which see theorem 5.3 of [Wil80] or page 255 of [MW84].

One expresses $U_{\mathfrak{p}}$ as a ratio of two Hecke operators which act via isogenies, and computes on the ordinary locus. □

4.3 Computations of cohomology classes

Suppose now that $k \geq 3$. Recall that we have associated an automorphic form π_1 on $G'(\mathbf{A_Q})$ to our Hilbert modular form π, together with a Galois representation $\overline{\rho}_f$. We write ϵ' for the prime-to-p part of the central character of π_1.

We now associate a regular differential (cf. section 8 of [Gro90]) ω_f on $\mathbf{M}'_{K'}$ to π_1, as in section 3.3. Write $a_\mathfrak{p}$ for the eigenvalue of $U_\mathfrak{p}$ on ω_f. Let ω_A denote the differential on $\mathbf{M}'_{Ig,H'}$ corresponding to $a^+ \otimes a^-$ under the Kodaira-Spencer isomorphism. We have an obvious action of $(\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*$ on $\mathbf{M}'_{Ig,H'}$, which we denote by $\langle\cdot\rangle$, so that $\langle\alpha\rangle\omega_f = \alpha^{k-2}\omega_f = \alpha^{-k'}\omega_f$, where $k' := p+1-k$. From the definitions of a^+ and a^- we see that $\langle\alpha\rangle a^+ = \alpha^{-1}a^+$ and $\langle\alpha\rangle a^- = \alpha^{-1}a^-$, so $\langle\alpha\rangle\omega_A = \alpha^{-2}\omega_A$.

Let v be a local parameter at a fixed supersingular point y on $\mathbf{M}'_{Ig,H'}$ such that $\langle\alpha\rangle v = \alpha^{-1}v$, so we may expand

$$\omega_A = \left(\sum_{n=1}^{\infty} c_n(v)v^{2+n(p-1)}\right)\frac{dv}{v}$$
$$\omega_f = \left(\sum_{n=0}^{\infty} a_n(v)v^{k'+n(p-1)}\right)\frac{dv}{v}.$$

Define an operator M on meromorphic differentials on $\mathbf{M}'_{Ig,H'}$ by

$$M\omega := d\left(\frac{\omega}{\omega_A}\right).$$

Then we have

Theorem 4.9. $\frac{M^{k'-1}\omega_f}{\omega_A} = \frac{(k'-1)!}{c_1(v)^{k'}}\left(a_0(v)v^{-pk'} + a_{k'}(v)v^{-k'} + \dots\right)$

Proof. Firstly, we note that $c_2(v) = 0$. This follows because, by Theorem 4.8, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} c_n(v)v^{2+n(p-1)}\right)\frac{dv}{v} &= \omega_A \\ &= U_\mathfrak{p}\omega_A \\ &= \left(\sum_{n=0}^{\infty} c_{pn+2}(v^\sigma)v^{2+n(p-1)}\right)\frac{dv}{v}.\end{aligned}$$

It is easy to see that the leading term of the right hand side of the proposed equality is correct, and this reduces us to computing $M^{k'}\omega_f$. If we use the relation $\frac{d}{dv}(v^p g) = v^p\frac{dg}{dv}$, we can turn the problem into one in characteristic zero; if we define formal power series

$$\omega'_A = \left(\sum_{n=1}^{\infty} C_n(v)v^{2-n}\right)\frac{dv}{v}$$
$$\omega'_f = \left(\sum_{n=0}^{\infty} A_n(v)v^{k'-n}\right)\frac{dv}{v},$$

with $C_2(v) = 0$ and an operation M' on differentials by

$$M'\omega := d\left(\frac{\omega}{\omega'_A}\right),$$

then it is enough to check that

$$M'^{k'}\omega'_f = \left(-\frac{k'!}{C_1(v)^{k'}}A_{k'}(v)v^{-k'} + \dots\right)\frac{dv}{v}.$$

We can easily convert our problem into one about power series, rather than differentials; define

$$\begin{aligned} D &= \left(\sum_{n=1}^{\infty} C_n(v)v^{1-n}\right)^{-1} \\ &= \frac{1}{vC_1(v)}\left(\sum_{n=0}^{\infty} D_n(v)v^{-n}\right), \end{aligned}$$

say, with $D_1(v) = 0$, and

$$f = \sum_{n=0}^{\infty} A_n(v)v^{k'-1-n}.$$

Then if we define an operator A on power series by $Ag = vD\frac{d}{dv}g$, it is not hard to see that it suffices to prove that the power series $D^{-1}A^{k'}Df$ has no terms in $v^{-1}, \dots, v^{1-k'}$. Then it suffices to prove that $A^{k'}v^j$ has no terms in $v^{-1}, \dots, v^{-k'}$ if $j \le k'$. We claim that we can write $v = cw + O(1)$ so that $Af = df/dw$; this is equivalent to $dw/dv = 1/(vD)$, and the integrability of $1/(vD)$ follows from its lack of a term in v^{-1}. Then as $v^j = O(w^j)$, we see that the k'-th derivative of v^j with respect to w is of the form constant + $O(w^{-k'+1})$ (where the constant is zero unless $j = k'$), as required. □

Corollary 4.10. $M^{k'}\omega_f = \left(-\frac{k'!}{c_1(v)^{k'}}a_{k'}(v)v^{-k'} + \dots\right)\frac{dv}{v}$.

Proof. This is immediate □

This result, showing that $M^{k'}\omega_f$ has a pole of low degree, allows us to associate a certain cohomology class to $M^{k'}\omega_f$. Let ss denote the divisor of supersingular points in $\mathbf{M}'_{Ig,H'}$, and let $U = \mathbf{M}'_{Ig,H'}\backslash ss$, and let η denote the generic point of $\mathbf{M}'_{Ig,H'}$.

Proposition 4.11. *The complex of groups*

$$\mathcal{O}_{\mathbf{M}'_{Ig,H'}}(U) \to \Omega^1_{\mathbf{M}'_{Ig,H'}/F^p_{\mathfrak{p}}} \oplus \mathcal{O}_{\mathbf{M}'_{Ig,H'},\eta}/\mathcal{O}_{\mathbf{M}'_{Ig,H'},ss} \\ \to \Omega^1_{\mathbf{M}'_{Ig,H'},\eta}/\Omega^1_{\mathbf{M}'_{Ig,H'}/F^p_{\mathfrak{p}}}(\log ss)_{ss},$$

where the first arrow takes h to (dh, h), and the second arrow takes a pair (ω, g) to $\omega - dg$, computes the de Rham cohomology of $\mathbf{M}'_{Ig,H'}$ with log poles on ss.

Proof. [CV92], §2. $\square$

Now, $M^{k'}\omega_f$ has poles of order at most $k'+1 = p+2-k \leq p-1$ on ss, by Corollary 4.10, so there is a section h of $\mathcal{O}_{\mathbf{M}'_{Ig,H'}}((p-1)ss)_{ss}$ such that $M^{k'}\omega_f - dh$ has at worst simple poles on ss. Such an h is well defined modulo $\mathcal{O}_{\mathbf{M}'_{Ig,H'},ss}$, so we have a well-defined cohomology class $[M^{k'}\omega_f] \in \mathbf{H}^1(\mathbf{M}'_{Ig,H'}, \Omega^\bullet_{\mathbf{M}'_{Ig,H'}}(\log ss))$. Furthermore, since $k \neq 2$ Corollary 4.10 shows that $M^{k'}\omega_f$ has zero residues, so in fact $[M^{k'}\omega_f] \in \mathbf{H}^1(\mathbf{M}'_{Ig,H'}, \Omega^\bullet_{\mathbf{M}'_{Ig,H'}})$.

Coleman has shown that there is an isomorphism between $\mathbf{H}^1(\mathbf{M}'_{Ig,H'}, \Omega^\bullet_{\mathbf{M}'_{Ig,H'}})$ and the quotient of the space of meromorphic differentials on $\mathbf{M}'_{Ig,H'}$ with no residues and poles of order at most p on $\mathbf{M}'_{Ig,H'}$ by the space of exact differentials dg, where g is a meromorphic function on $\mathbf{M}'_{Ig,H'}$ with poles of order at most $(p-1)$ (for a statement, see page 499 of [Gro90]; the proof is a straightforward exercise in Čech cohomology). Thus if $[M^{k'}\omega_f] = 0$ there is a meromorphic function h on $\mathbf{M}'_{Ig,H'}$ with poles only at the supersingular points, of order at most $(p-1)$, satisfying $dh = M^{k'}\omega_f$. Furthermore, replacing h by an element of the vector space spanned by $\langle h, U_{\mathfrak{p}}h, \dots \rangle$, we see that there is a meromorphic function h with poles of order at most $(p-2)$ on $\mathbf{M}'_{Ig,H'}$, satisfying $U_{\mathfrak{p}}h = b_{\mathfrak{p}}h$ for some $b_{\mathfrak{p}}$.

Lemma 4.12. *$T_{\mathfrak{l}}h = a_{\mathfrak{l}}h$.*

Proof. We have $dh = M^{k'}\omega_f$, so $M(h\omega_A) = M^{k'}\omega_f$. Thus by Theorem 4.6 we have have

$$\begin{aligned}
M(T_{\mathfrak{l}}(h\omega_A)) &= T_{\mathfrak{l}}(M(h\omega_A)) \\
&= T_{\mathfrak{l}}(M^{k'}\omega_f) \\
&= M^{k'}(T_{\mathfrak{l}}\omega_f) \\
&= a_{\mathfrak{l}}M^{k'}\omega_f \\
&= a_{\mathfrak{l}}M(h\omega_A)
\end{aligned}$$

so $M(T_{\mathfrak{l}}(h\omega_A) - a_{\mathfrak{l}}h\omega_A) = 0$, and $T_{\mathfrak{l}}(h\omega_A) - a_{\mathfrak{l}}h\omega_A = g^p\omega_A$ for some g. But the equation $T_{\mathfrak{l}}h - a_{\mathfrak{l}}h = g^p$ immediately implies that $g = 0$ unless $k = p$, upon comparison of leading terms. If $k = p$, then we use the fact that $T_{\mathfrak{l}}h - a_{\mathfrak{l}}h$ is an eigenform for $U_{\mathfrak{p}}$, with eigenvalue $b_{\mathfrak{p}}$. Then $b_{\mathfrak{p}}g^p = U_{\mathfrak{p}}g^p = g$, which again gives $g = 0$. $\square$

From Corollary 4.10 we have

$$\langle \alpha \rangle h = \alpha^{-k'}h.$$

Then the holomorphic differential $\omega_{f'} = \omega_A h$ satisfies

$$T_{\mathfrak{l}}\omega_{f'} = a_{\mathfrak{l}}\omega_{f'}$$
$$\langle\alpha\rangle\omega_{f'} = \alpha^{-k'}\omega_{f'}$$
$$U_{\mathfrak{p}}\omega_{f'} = b_{\mathfrak{p}}\omega_{f'}$$

for some $b_{\mathfrak{p}}$.

Thus we have:

Theorem 4.13. *If $[M^{k'}\omega_f] = 0$, then there is an ordinary "companion form" $\omega_{f'}$. The representation $\overline{\rho}_{f'}$ attached to $\omega_{f'}$ is isomorphic to $\overline{\rho}_f$.*

Proof. We have everything except for the assertion that $\overline{\rho}_{f'} \cong \overline{\rho}_f$, and the claim that $b_{\mathfrak{p}} \neq 0$. The first follows at once from the Cebotarev density theorem and the fact that $T_{\mathfrak{l}}\omega_{f'} = a_{\mathfrak{l}}\omega_{f'}$. To prove that $b_{\mathfrak{p}} \neq 0$, note simply that if $b_{\mathfrak{p}} = 0$ then $\omega_{f'} = M^{k'-1}\omega_f$, which is nonsense (from a comparison of leading terms). □

We must now show that if $\overline{\rho}_f$ is unramified at $\mathfrak{p}$, then $[M^{k'}\omega_f] = 0$.

Theorem 4.14. $\sigma^{-1}\operatorname{Frob}[M^{k'}\omega_f] = \frac{a_{\mathfrak{p}}}{\epsilon(\mathfrak{p})}[M^{k'}\omega_f]$.

Proof. The proof of this is similar to the proof of Theorem 5.1 of [CV92] (note however that our proof works for all $3 \leq k \leq p$, and may be used to replace the appeal to rigid analysis in [CV92]).

By Corollary 4.10, $[M^{k'}\omega_f]$ is represented by the cocycle

$$\left(M^{k'}\omega_f, \frac{(k'-1)!}{c_1(v)^{k'}}a_{k'}(v)v^{-k'}\right).$$

Since $M^{k'}\omega_f = d(M^{k'-1}\omega_f/\omega_A)$, this class is also represented by

$$\left(0, -\frac{(k'-1)!}{c_1(v)^{k'}}a_0(v)v^{-pk'}\right).$$

Then if ϕ is the Frobenius endomorphism of $\mathbf{M}'_{Ig,H'}$, $\sigma^{-1}\operatorname{Frob}[M^{k'}\omega_f]$ is represented by

$$\phi^*\left(M^{k'}\omega_f, \frac{(k'-1)!}{c_1(v)^{k'}}a_{k'}(v)v^{-k'}\right) = \left(0, \frac{(k'-1)!}{c_1(v^\sigma)^{k'}}a_{k'}(v^\sigma)v^{-pk'}\right).$$

We thus need to demonstrate the equality

$$a_{k'}(v^\sigma)c_1(v)^{k'} = -\frac{a_{\mathfrak{p}}}{\epsilon(\mathfrak{p})}a_0(v)c_1(v^\sigma)^{k'}.$$

The equality $U_{\mathfrak{p}}\omega_f = a_{\mathfrak{p}}\omega_f$ yields $a_{k'}(v) = a_{\mathfrak{p}}a_0(v^\sigma)$ and thus $a_{k'}(v^\sigma) = a_{\mathfrak{p}}a_0(v^{\sigma^2})$. A straightforward computation shows that on the supersingular

locus we have the equality $\sigma^{-2} = \langle\mathfrak{p}\rangle \cdot \langle -1\rangle_{\mathfrak{p}}$, where $\langle\mathfrak{p}\rangle$ is the Hecke operator $\left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & \mathfrak{p}\end{smallmatrix}\right)$ and $\langle -1\rangle_{\mathfrak{p}}$ is given by

$$\langle -1\rangle_{\mathfrak{p}} : (A, \iota, \lambda, \eta_{\mathfrak{p}}, \overline{\eta}_{\mathfrak{p}}, P, Q) \mapsto (A, \iota, \lambda, \eta_{\mathfrak{p}}, \overline{\eta}_{\mathfrak{p}}, -P, -Q).$$

We thus have $(\omega_f)^{\sigma^{-2}} = -\epsilon(\mathfrak{p})\omega_f$, from which we obtain $a_0(v^{\sigma^2})(v^{\sigma^2}|\langle\mathfrak{p}\rangle^{-1}\langle -1\rangle_{\mathfrak{p}}^{-1})^{k'} = -\frac{1}{\epsilon(\mathfrak{p})}a_0(v)v^{k'}$. Combining these results, we have

$$\begin{aligned} a_{k'}(v^{\sigma})c_1(v)^{k'} &= a_{\mathfrak{p}}a_0(v^{\sigma^2})v^{k'}c_1(v)^{k'} \\ &= -\frac{a_{\mathfrak{p}}}{\epsilon(\mathfrak{p})}a_0(v)\left(\left(\frac{v}{v^{\sigma^2}|\langle\mathfrak{p}\rangle^{-1}\langle -1\rangle_{\mathfrak{p}}^{-1}}\right)(y)c_1(v)\right)^{k'}, \end{aligned}$$

and the result follows from Lemma 4.15. $\square$

Lemma 4.15. $\left(\frac{v}{v^{\sigma^2}|\langle\mathfrak{p}\rangle^{-1}\langle -1\rangle_{\mathfrak{p}}^{-1}}\right)(y) = c_1(v^{\sigma})/c_1(v)$.

Proof. This may be proved in exactly the same fashion as Lemma 5.3 of [CV92], except that rather than working with the canonical elliptic curve over $\mathbf{F}_{p^2}$ one works with the canonical formal $\mathcal{O}_{\mathfrak{p}}$-module defined over $\mathbf{F}_{q^2}$, for which see Proposition 1.7 of [Dri76]. $\square$

Definition 4.16. As in Corollary 3.10, let ω_F be a regular differential on $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$ which is an eigenform for the Hecke operators $T_{\mathfrak{l}}$, $U_{\mathfrak{p}}$, and $\langle\alpha\rangle$, with eigenvalues lifting those of ω_f. We also write ω_F for the corresponding differential on $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$.

Definition 4.17. Define a Hecke operator $w_{\mathfrak{p}}$ on $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$ by $\left(\begin{smallmatrix}0 & 1\\ -\mathfrak{p} & 0\end{smallmatrix}\right)$. This gives a Hecke operator $w_{\mathfrak{p}}$ on $\mathbf{M}'_{bal.U_1^p(\mathfrak{p}),H'}$ by descent.

Theorem 4.18. *Let X' be the subscheme of $\mathbf{M}'_{bal.U_1^p(\mathfrak{p}),H'}$ obtained by removing the closed subscheme $\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'}$. Then $(\omega_F|U_{\mathfrak{p}}w_{\mathfrak{p}})|_{X'}$ is divisible by $\pi^{k'}$, so that there is a cohomology class $[(\omega_F|U_{\mathfrak{p}}w_{\mathfrak{p}})|_{X'}/\pi^{k'}]$ on $\mathbf{M}'_{Ig,H'}$. Then*

$$[(\omega_F|U_{\mathfrak{p}}w_{\mathfrak{p}})|_{X'}/\pi^{k'}] = -u\frac{\epsilon(\mathfrak{p})}{k'!}[M^{k'}\omega_f],$$

for some unit u.

Proof. Recall that the completed local ring of $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$ at a supersingular point is $\mathcal{O}_{F^0_{\mathfrak{p}}}[[v,w]]/(vw-\pi)$, where $\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'}$ is given by $w = 0$. Thus on X' the function w is invertible, and we have $v = \pi/w$. An easy check shows that on the supersingular locus we have $w_{\mathfrak{p}} = \sigma \cdot \langle\mathfrak{p}\rangle$, whence we have

$$
\begin{aligned}
(\omega_F|U_{\mathfrak{p}}w_{\mathfrak{p}})|_{X'} &= \left(\epsilon(\mathfrak{p})a_{k'}(w)(w^\sigma)^{k'} + \dots\right)\frac{dv}{v} \bmod p \\
&= \left(u\epsilon(\mathfrak{p})a_{k'}(v)\frac{\pi^{k'}}{v^{k'}c_1(v)^{k'}} + \dots\right)\frac{dv}{v} \bmod p,
\end{aligned}
$$

where we have used the identity $c_1(v)^{-1} = uv(v^\sigma|w_{\mathfrak{p}}\langle\mathfrak{p}\rangle^{-1})$, which may again be proved in exactly the same fashion as Lemma 5.3 of [CV92], except that rather than working with the canonical elliptic curve over $\mathbf{F}_{p^2}$ one works with the canonical formal $\mathcal{O}_{\mathfrak{p}}$-module defined over $\mathbf{F}_{q^2}$, for which see Proposition 1.7 of [Dri76].

So $(\omega_F|U_{\mathfrak{p}}w_{\mathfrak{p}})|_{X'}$ is divisible by $\pi^{k'}$, and comparing this with Corollary 4.10 we see that the claim follows.

□

4.4 Pairings

We now recall from [CV92] the relationship between the Kodaira-Spencer and Serre-Tate pairings, and a formula relating the Kodaira-Spencer pairing on a semi-stable curve to the cup product of certain (log-) cohomology classes.

Let R be a complete local ring with residue field $\mathbf{F}$ of characteristic p, and let $W(\mathbf{F})$ denote the ring of Witt vectors of $\mathbf{F}$. Let G be a p-divisible group over R with dual $G^\vee$, and let Ω_G, $\Omega_{G^\vee}$ denote the invariant one-forms on G/R, $G^\vee/R$ respectively. Then (see Corollary 4.8.iii of [Ill85]) there is a pairing, functorial for morphisms of p-divisible groups over R,

$$\kappa : \Omega_G \otimes \Omega_{G^\vee} \to \Omega^1_{R/W(\mathbf{F})}.$$

Let $\overline{G}$, $\overline{G}^\vee$ denote the special fibres of G, $G^\vee$ respectively. We have the Serre-Tate pairing (see [Kat81])

$$q : T_p\overline{G} \times T_p\overline{G}^\vee \to 1 + m_R,$$

where T_p denotes the p-adic Tate module, and m_R is the maximal ideal of R. We can view an element $\alpha \in T_pG$ as a homomorphism from $G^\vee$ to $\mathbf{G}_m$, and we define $\omega_\alpha = \alpha^*(dt/t) \in \Omega_{G^\vee}$. If $\alpha^\vee \in T_pG^\vee$ we define $\omega_{\alpha^\vee} \in \Omega_G$ in the same fashion. For $a \in R^*$, let $d\log(a) = d_{R/W(\mathbf{F})}a/a \in \Omega^1_{R/W(\mathbf{F})}$; then we have

Theorem 4.19. *Suppose that G is an ordinary p-divisible group over R (that is, suppose that the dual of the connected subgroup of $\overline{G}$ is étale). Then for any $\alpha \in T_pG$, $\alpha^\vee \in T_pG^\vee$, we have*

$$d\log q(\alpha, \alpha^\vee) = \kappa(\omega_{\alpha^\vee} \otimes \omega_\alpha).$$

Proof. This is Theorem 1.1 of [CV92]. $\square$

Let $S = \mathbf{F}[t]/(t^{b+1})$ with $0 \leq b < p$. Let $S^\times$ denote the log-scheme associated to the pre-log structure $\mathbf{N} \to S$, $1 \mapsto t$. Let M_S denote the corresponding monoid, with an element T mapping to t. Let $\mathbf{F}$ (slightly abusively) denote $\mathbf{F}$ with the trivial log-structure; it follows easily that $\Omega^1_{S^\times/\mathbf{F}}$ is a free S-module generated by $d\log T$.

Let $s : X \to \mathrm{Spec}(R)$ be a semi-stable curve over S (that is, X is locally isomorphic to $xy = t$ in the étale topology), and suppose that there is a lifting $\tilde{X}$ of X to a semi-stable curve over $\tilde{S}$, where $\tilde{S}$ is a discrete valuation ring with $\tilde{S}$ mod $p = S$ and the generic fibre of $\tilde{X}$ is smooth over the generic point of $\tilde{S}$ (this will, of course, hold in our applications to PEL Shimura curves). We have natural log-structures on $\tilde{X}$ and $\tilde{S}$ given by the subsheaf of the structure sheaf whose sections become invertible upon removal of the special fibre. Let $X^\times$ be the reduction of this log-scheme to S; $X^\times$ is smooth over $S^\times$.

We have an exact sequence of sheaves (see Proposition 3.12 of [Kat89])

$$0 \to s^*\Omega^1_{S^\times/\mathbf{F}} \to \Omega^1_{X^\times/\mathbf{F}} \to \Omega^1_{X^\times/S^\times} \to 0.$$

Let $Kod : \mathrm{H}^0(X, \Omega^1_{X^\times/S^\times}) \to \mathrm{H}^1(X, s^*\Omega^1_{S^\times/\mathbf{F}}) \cong \mathrm{H}^1(X, \mathcal{O}_X) \otimes \Omega^1_{S^\times/\mathbf{F}}$ denote the boundary map in the corresponding long exact sequence of cohomology. When X is smooth over S there is a natural isomorphism $\Omega^1_{X^\times/S^\times} \xrightarrow{\sim} \Omega^1_{X/S}$, and Kod is the composition of the usual Kodaira-Spencer map and the natural map $\mathrm{H}^1(X, \mathcal{O}_X) \otimes \Omega^1_{S/\mathbf{F}} \to \mathrm{H}^1(X, \mathcal{O}_X) \otimes \Omega^1_{S^\times/\mathbf{F}}$.

Suppose that the reduction of X mod t is $\overline{X} = C_1 \cup C_2$ with C_1, C_2 smooth irreducible curves. Let $D = C_1 \cap C_2$, $U_1 = X - C_2$, $U_2 = X - C_1$. Let d denote the boundary map for the complex $\Omega^\bullet_{X^\times/S^\times}$. Then we have:

Theorem 4.20. *Suppose ω is in the image of the natural map from $\mathrm{H}^0(X, \Omega^1_{X/S})$ to $\mathrm{H}^0(X, \Omega^1_{X^\times/S^\times})$ and that $\omega|_{U_2} = t^b\eta$ for $\eta \in \Omega^1_{X/S}(U_2)$. Then*

$$\overline{\eta} := \eta|_{C_2} \in \mathrm{H}^0(C_2, \Omega^1_{C_2/\mathbf{F}}((b+1)D)).$$

In particular, we obtain a cohomology class

$$[\overline{\eta}] \in \mathbf{H}^1(C_2, \Omega^1_{C_2/\mathbf{F}}(\log D))$$

in the same fashion as in Section 4.3. If ν is in the image of $\mathrm{H}^0(X, \Omega^1_{X/S})$ in $\mathrm{H}^0(X, \Omega^1_{X^\times/S^\times})$ and $\nu|_{U_1} \in t^b\Omega^1_{X^\times/S^\times}(U_1)$, then

$$\nu \cdot Kod(\omega) = ([\nu|_{C_2}], [\eta|_{C_2}])_{C_2} bt^b d\log T,$$

where $(,)_{C_2}$ is the usual cup product pairing.

Proof. This is Theorem 3.1 of [CV92]. $\square$

4.5 The local Galois Representation at $\mathfrak{p}$

In this section we examine the local Galois representation on the Jacobian of $\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}$. We write $J := \mathrm{Jac}(\mathbf{M}'^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'})$, $\mathbf{T} \subseteq \mathrm{End}_{\mathbf{Q}}(J)$ generated by the $U_\mathfrak{p}$, $T_\mathfrak{l}$, $S_\mathfrak{l}$ for $\mathfrak{l}$ unramified and not in the level, and the operators $\langle\alpha\rangle$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$ above the (minimal prime) ideal corresponding to ω_f. $\mathbf{T}$ is free of finite rank over $\mathbf{Z}$, so the ring $\mathbf{T}_p = \varprojlim \mathbf{T}/p^n\mathbf{T} = \mathbf{T}\otimes\mathbf{Z}_p$ is a complete semilocal $\mathbf{Z}_p$-algebra of finite rank. The ring $\mathbf{T}_\mathfrak{m} = \varprojlim \mathbf{T}/\mathfrak{m}^n\mathbf{T}$ is complete and local, and by the theory of complete semilocal rings $\mathbf{T}_\mathfrak{m}$ is a direct factor of $\mathbf{T}_p$, so we have an idempotent decomposition of the identity

$$\mathbf{T}_p = \mathbf{T}_\mathfrak{m} \times \mathbf{T}'_\mathfrak{m}$$
$$1 = \epsilon_\mathfrak{m} + \epsilon_{\mathfrak{m}'}.$$

Let $h = \mathrm{rk}_{\mathbf{Z}_p}(\mathbf{T}_\mathfrak{m})$. Let G be the p-divisible group over E defined by $T_pG = \epsilon_\mathfrak{m}T_pJ$.

Theorem 4.21. *1. The p-divisible group G has height $2h$ and is isomorphic to $G^\vee$ over $E_{\mathfrak{n}\mathfrak{p}}$, where $E_{\mathfrak{n}\mathfrak{p}}$ is the ray class field of conductor $\mathfrak{n}\mathfrak{p}$. It has good reduction over $F'_\mathfrak{p}$, where $F'_\mathfrak{p}$ is the tame extension of $F_\mathfrak{p}$ of degree $q-1$ corresponding to our choice of uniformiser $\mathfrak{p}$.*

2. Let $\overline{G}$ be the reduction of G over $\mathcal{O}_{F'_\mathfrak{p}}/\mathfrak{m}_{\mathcal{O}_{F'_\mathfrak{p}}}\mathcal{O}_{F'_\mathfrak{p}} = \mathbf{F}_p$, and $D(\overline{G})$ be its Dieudonné module. Then $\overline{G} = \overline{G}^m \times \overline{G}^e$, where $\overline{G}^m$ is multiplicative and $\overline{G}^e$ is étale, and $\overline{G}^m$ and $\overline{G}^e$ both have height h over $\mathbf{F}_p$. The endomorphisms F, V commute with the action of $\mathbf{T}_\mathfrak{m}$, F acts on $D(\overline{G}^e)$ by multiplication by the unit $U_\mathfrak{p}\cdot\langle\mathfrak{p}\rangle^{-1}$ of $\mathbf{T}_\mathfrak{m}$, and V acts by multiplication by the unit $U_\mathfrak{p}$ of $\mathbf{T}_\mathfrak{m}$.

3. The exact sequence $0 \to G^0 \to G \to G^e \to 0$ of p-divisible groups over $F^1_\mathfrak{p}$ gives a $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$-stable filtration $0 \to T_pG^0 \to T_pG \to T_pG^e \to 0$. $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ acts on T_pG^0 by the character $\lambda(U_\mathfrak{p}^{-1})\cdot\chi$, where χ is the cyclotomic character and $\lambda(\cdot)$ is the unramified character sending $\mathrm{Frob}_\mathfrak{p}$ to $\cdot$, and on T_pG^e by the character $\lambda(U_\mathfrak{p}\cdot\langle\mathfrak{p}\rangle^{-1})\cdot\chi^{2-k}$. Thus there is a short exact sequence $0 \to G^0[\mathfrak{m}] \to G[\mathfrak{m}] \to G^e[\mathfrak{m}] \to 0$ over $\mathbf{Q}_p$, with flat extensions to $\mathcal{O}_\mathfrak{p}[\alpha]$, where $G^e[\mathfrak{m}]$ does not necessarily denote the full $\mathfrak{m}$-torsion in G^e, but rather the cokernel of the map $G^0[\mathfrak{m}] \to G[\mathfrak{m}]$. The Galois group $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ acts on $G^0[\mathfrak{m}]$ via the character $\lambda(1/a_\mathfrak{p})\cdot\chi$ and on $G^e[\mathfrak{m}]$ via the character $\lambda(a_\mathfrak{p}/\epsilon'(\mathfrak{p}))\cdot\chi^{2-k}$.

Proof. 1. The height of G is equal to the dimension of $V_pG = T_pG\otimes\mathbf{Q}_p$ as a $\mathbf{Q}_p$-vector space. But by Lemma 3.8 V_pG is a free $\mathbf{T}_\mathfrak{m}\otimes\mathbf{Q}_p$-module of rank 2, so the height G is $2h$.

That G is isomorphic to $G^\vee$ over $E_{\mathfrak{n}\mathfrak{p}}$ follows from the existence of a nondegenerate alternating form $\langle,\rangle : T_pG \times T_pG \to T_p\mathbf{G}_m$ satisfying

$$\langle a^{\sigma_\mathfrak{l}}, b^{\sigma_\mathfrak{l}}\rangle = \frac{\mathbf{N}\mathfrak{l}}{\psi_{\pi'}(\mathfrak{l})}\langle a, b\rangle$$

where $\sigma_{\mathfrak{l}}$ is a Frobenius element at $\mathfrak{l}$. The existence of such a form can either be deduced as in §11 of [Gro90] by modifying the Weil pairing on J by an Atkin-Lehner involution, or by (somewhat perversely) deducing its existence from Theorem 3.9. Then if $\mathfrak{l}$ is trivial in the ray class group mod $\mathfrak{np}$ we have $\langle a^{\sigma_{\mathfrak{l}}}, b^{\sigma_{\mathfrak{l}}}\rangle = \mathbf{N}\mathfrak{l}\langle a, b\rangle$; but such $\sigma_{\mathfrak{l}}$ are dense in $\mathrm{Gal}(\overline{E}/E_{\mathfrak{np}})$, so G is isomorphic to $G^{\vee}$ over $E_{\mathfrak{np}}$.

The proof that G has good reduction over $F'_{\mathfrak{p}}$ is very similar to the argument in the proof of Proposition 12.9.1 in [Gro90]. Let A be the connected subgroup of points P in J with $\sum_{a\in(\mathcal{O}_{\mathfrak{p}})^*}\langle a\rangle P = 0$, and B the connected subgroup of points fixed by the action of the group $\{\langle a\rangle : a \in (\mathcal{O}_{\mathfrak{p}})^*\}$. The isogeny

$$\phi : J \to A \times B$$
$$P \mapsto \left((q-1)P - \sum_{a\in(\mathcal{O}_{\mathfrak{p}})^*}\langle a\rangle P, \sum_{a\in(\mathcal{O}_{\mathfrak{p}})^*}\langle a\rangle P\right)$$

has degree prime to p, and thus induces an isomorphism on p-divisible groups, because the composite with the natural injection $A\times B \hookrightarrow J$ is just $J \xrightarrow{q-1} J$. Because $k \neq 2$, the p-divisible group of G is a subgroup of the p-divisible group of A, so it suffices to prove that A has good reduction over $F'_{\mathfrak{p}}$. This follows from the arguments of [DR73] I.3.7 and V.3.2.

2. This follows from knowledge of the action of $U_{\mathfrak{p}}$ on the components $\mathbf{M}'_{Ig,H'}$ and $\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'}$ of the special fibre $\mathbf{M}'_{bal.U_1(\mathfrak{p});det=1,H'}$. As in Proposition 12.9.2 of [Gro90] we have

$$\overline{G} \cong \lim_{\leftarrow} \mathrm{Jac}(\mathbf{M}'_{Ig,H'})[\mathfrak{m}^n] \times \mathrm{Jac}(\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'})[\mathfrak{m}^n].$$

We claim that in fact we have $\overline{G}^m = \lim_{\leftarrow} \mathrm{Jac}(\mathbf{M}'_{Ig,H'})[\mathfrak{m}^n]$ and $\overline{G}^e = \lim_{\leftarrow} \mathrm{Jac}(\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'})[\mathfrak{m}^n]$. Indeed, $U_{\mathfrak{p}}$ is a unit in $\mathbf{T}_{\mathfrak{m}}$, and acts as Ver on $\mathbf{M}'_{Ig,H'}$ (note that this Ver is the dual of the p-power Frobenius), whence $\mathrm{Jac}(\mathbf{M}'_{Ig,H'})[\mathfrak{m}^n]$ is multiplicative (by the standard theory of Dieudonné modules). Similarly, one can check that $U_{\mathfrak{p}}$ acts as $\mathrm{Frob}\cdot\langle\mathfrak{p}\rangle$ on $\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'}$, so $\mathrm{Jac}(\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'})[\mathfrak{m}^n]$ is étale. That both subgroups have height h follows from the self-duality of $\overline{G}$ over $E_{\mathfrak{np}}$.

3. The filtration $0 \to T_pG^0 \to T_pG \to T_pG^e \to 0$ is stable under the action of $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F'_{\mathfrak{p}})$, which acts (by the above) via the characters $\lambda(U_{\mathfrak{p}}^{-1})\cdot\chi$ on T_pG^0 and $\lambda(U_{\mathfrak{p}})$ on T_pG^e. These characters are nonconjugate, so the filtration is in fact stable under the action of $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$. To determine this action it suffices to compute the action of $\mathrm{Gal}(F'_{\mathfrak{p}}/F_{\mathfrak{p}})$;

but this may be accomplished just as in [Gro90]. It is easy to check that $\mathrm{Gal}(F'_{\mathfrak{p}}/F_{\mathfrak{p}})$ acts trivially on $\mathbf{M}'_{Ig,H'}$ and by $\langle\alpha\rangle$ on $\mathbf{M}'^{(\sigma^{-1})}_{Ig,H'}$, so by χ^{2-k} on $\overline{G}^e$, as required. The existence of the short exact sequence

$$0 \to G^0[\mathfrak{m}] \to G[\mathfrak{m}] \to G^e[\mathfrak{m}] \to 0$$

follows at once, and the action of $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ comes from the observation that it acts semisimply by the results of [BLR91].

□

Corollary 4.22. *Let W be the two-dimensional vector space underlying the representation $\overline{\rho}_f$. Then there is a short exact sequence of* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$*-modules $0 \to V \to W \to V' \to 0$. The group* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ *acts on V by the character $\chi^{k-1}\lambda(\epsilon'(\mathfrak{p})/a_{\mathfrak{p}})$ and on V' by the unramified character $\lambda(a_{\mathfrak{p}})$. Equivalently, there is a basis for W such that* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ *acts via the upper triangular matrices*

$$\begin{pmatrix} \chi^{k-1}\lambda(\epsilon'(\mathfrak{p})/a_{\mathfrak{p}}) & * \\ 0 & \lambda(a_{\mathfrak{p}}) \end{pmatrix}.$$

Proof. This is immediate. □

Definition 4.23. Let L be the field of definition of the representation $\overline{\rho}_f$: $\mathrm{GL}_2(\overline{E}/E) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ i.e. the smallest field $L/\mathbf{F}_p$ such that $\overline{\rho}_f$ factors through $GL_2(L)$.

We have a realisation of $\overline{\rho}_f \otimes (\epsilon'\chi^{k-2})^{-1}$ on $G[\mathfrak{m}]$. The L-vector space scheme $G[\mathfrak{m}]$ sits in a short exact sequence over $F_{\mathfrak{p}}$

$$0 \to G^0[\mathfrak{m}] \to G[\mathfrak{m}] \to G^e[\mathfrak{m}] \to 0 \qquad (\star)$$

of L-vector space schemes, with $G^0[\mathfrak{m}]$ and $G^e[\mathfrak{m}]$ both one-dimensional.

By their definitions, the L-vector space schemes in $(\star)$ all have canonical extensions to $\mathcal{O}_{F^1_{\mathfrak{p}}}$. In order to determine when $\overline{\rho}_f$ is tamely ramified, we will determine when $(\star)$ splits; in fact:

Lemma 4.24. *The following are equivalent:*

- *The sequence of L-vector space schemes in $(\star)$ is uniquely split over $F_{\mathfrak{p}}$.*
- *The sequence of L-vector space schemes over $\mathcal{O}_{F^1_{\mathfrak{p}}}$ which extends $(\star)$ is uniquely split over $\mathcal{O}_{F^1_{\mathfrak{p}}}$.*
- *The restriction of $\overline{\rho}_f$ to* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ *is diagonalisable, and the sum of distinct characters $\chi^{k-1}\lambda(\epsilon'(\mathfrak{p})/a_{\mathfrak{p}})$ and $\lambda(a_{\mathfrak{p}})$.*

Proof. The equivalence of (1) and (3) is immediate. The extension $F^1_{\mathfrak{p}}/F_{\mathfrak{p}}$ has degree prime to p, so (1) is equivalent to the splitting of ($\star$) over $F^1_{\mathfrak{p}}$, which is obviously implied by (2). To establish that (1) implies (2), we may base change to an étale extension R of degree prime to p of $\mathcal{O}_{F^1_{\mathfrak{p}}}$, and check that a splitting of ($\star$) over the quotient field S of R implies a splitting over R.

Choose R so that $\lambda(1/a_{\mathfrak{p}})$, $\lambda(a_{\mathfrak{p}}/\epsilon'(\mathfrak{p}))$ are trivial on $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/S)$. Then, as in the proof of Proposition 13.2 of [Gro90], the L-vector space scheme $G^e[\mathfrak{m}]$ is isomorphic to the étale vector space scheme $L = L \otimes (\mathbf{Z}/p\mathbf{Z})$ with trivial Galois action over R, and $G^0[\mathfrak{m}]$ is isomorphic to the Cartier dual $L^t = L^\vee \otimes \mu_p$ over R, where $L^\vee = \mathrm{Hom}(L, \mathbf{Z}/p\mathbf{Z})$. But we have a Kummer-theoretic canonical isomorphism of L-vector spaces

$$\mathrm{Ext}_R(L, L^\vee \otimes \mu_p) \xrightarrow{\sim} R^*/R^{*p} \otimes L^\vee$$

where Ext_R classifies extensions in the category of L-vector space schemes.

Thus the sequence ($\star$) over R gives a class in $R^*/R^{*p} \otimes L^\vee$ which is zero if and only if ($\star$) splits; but R^*/R^{*p} injects into $S^*/S^{*p} \otimes L^\vee$, so a splitting over S implies one over R. $\square$

Now let R denote the completion of the ring of integers in the maximal unramified extension of $\mathcal{O}_{F^1_{\mathfrak{p}}}$. We now define a bilinear pairing

$$q_f : (B^t)^e(L) \times B^e(L) \to (R^*/R^{*p}) \otimes L^\vee$$

where $B := G[\mathfrak{m}] \otimes (\epsilon' \otimes \chi^{k-2})$, just as in §6 of [CV92]. An element of $(B^t)^e(L) \times B^e(L)$ corresponds to a pair of homomorphisms $\alpha : G^0[\mathfrak{m}] \to L^\vee \otimes \mu_p$ and $\beta : L \to G^e[\mathfrak{m}]$, which give (via push-out and pull-back) a homomorphism $\alpha_*\beta^* : \mathrm{Ext}_R(G^e[\mathfrak{m}], G^0[\mathfrak{m}]) \to \mathrm{Ext}_R(L, L^\vee \otimes \mu_p) \xrightarrow{\sim} R^*/R^{*p} \otimes L^\vee$, and the required element of $R^*/R^{*p} \otimes L^\vee$ is the image of the extension class of $G[\mathfrak{m}]$ under $\alpha_*\beta^*$.

We have a map $\mathrm{tr}^\vee : L^\vee \to \mathbf{F}_p$ given by $h \mapsto h(1)$, and a map $d\log : R^* \to \Omega^1_{R/\mathbf{Z}_p}$ given by $a \mapsto da/a$, and thus a pairing $d\log q_f := (d\log \otimes \mathrm{tr}^\vee) \circ q_f : (B^t)^e(L) \times B^e(L) \to \Omega^1_{R/\mathbf{Z}_p}$.

Lemma 4.25. *If $\overline{\rho}_f$ is tamely ramified at $\mathfrak{p}$, then the pairing $d\log q_f$ is trivial.*

Proof. It suffices to show that q_f is trivial; but as noted above q_f is trivial if and only if ($\star$) splits if and only if $\overline{\rho}_f|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ is diagonalisable if and only if $\overline{\rho}_f$ is tamely ramified above $\mathfrak{p}$. $\square$

Since the p-divisible group B is ordinary, we have (see §4.4) the Serre-Tate pairing

$$q : T\overline{B} \times T\overline{B}^t \to 1 + \pi R$$

and thus a pairing

$$d\log q : T\overline{B} \times T\overline{B}^t \to \Omega^1_{R/\mathcal{O}_{\mathfrak{p}}^{nr}}$$

which extends by scalars to a pairing

$$d\log q : (T\overline{B}(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} R) \times (T\overline{B}^t(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} R) \to \Omega^1_{R/\mathcal{O}_{\mathfrak{p}}^{nr}}.$$

Let $\overline{G}(j)$ denote the subgroup on which $(\mathcal{O}/\mathfrak{p})^*$ acts via $\langle\alpha\rangle = \alpha^j$. Then we have:

Theorem 4.26. *If $\alpha \in T\overline{B}(-k') \otimes R$, $\beta \in T\overline{B}^t(k') \otimes R$, and $\omega_\alpha|_{\mathbf{M}'_{Ig,H'}} = \omega_f$ (where $\omega_\alpha = \alpha^*(dt/t)$), then*

$$d\log q(\alpha,\beta) = \frac{u}{(k'-1)!a_{\mathfrak{p}}}\Big((w_{\mathfrak{p}}^*\omega_\beta)|\mathbf{M}'_{Ig,H'}, [M^{k'}\omega_f]\Big)_{\mathbf{M}'_{Ig,H'}} \pi^{k'-1}d\pi + \dots$$

where u is as in Theorem 4.18.

Proof. This is very similar to Theorem 4.4 of [CV92]. From Theorem 4.18 we have $[(\omega_f|U_{\mathfrak{p}}w_{\mathfrak{p}})|_{X'}/\pi^{k'}] = -\frac{\epsilon'(\mathfrak{p})}{k'!}[M^{k'}\omega_f]$, and $\omega_\alpha|U_{\mathfrak{p}} = a_{\mathfrak{p}}\omega_\alpha \bmod \pi$, so the result follows from Theorem 4.9. □

The Serre-Tate pairing $q : B[p](\overline{\mathbf{F}_p}) \times B^t[p](\overline{\mathbf{F}_p}) \to R^*/R^{*p}$ is related to q_f as follows:

Lemma 4.27. *Suppose that $\alpha \in B[p](\overline{\mathbf{F}}_p)$ and $\beta \in B(\overline{\mathbf{F}}_p)$. Then*

$$(1 \otimes \mathrm{tr}^\vee)q_f(\alpha \textit{ mod } \mathfrak{m}, \beta) = q(\alpha,\beta) \textit{ mod } R^{*p}.$$

Proof. This follows from the definitions of q_f, q. □

We have a map

$$T(B^e) \to \mathrm{H}^0(\mathbf{M'}^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}, \Omega^1_{\mathbf{M'}^{\mathrm{can}}_{bal.U_1(\mathfrak{p}),H'}/R}) \quad \to \mathrm{H}^0(\mathbf{M}'_{Ig,H'}, \Omega^1_{\mathbf{M}'_{Ig,H'}/\overline{\mathbf{F}}_p}).$$

Let β_f be the element of $B^e(\overline{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$ corresponding to ω_f via this map. Then

Proposition 4.28. *$[M^{k'}\omega_f] = 0$ if and only if $d\log q_f(\alpha,\beta_f) = 0$ for all $\alpha \in (B^t)^e(\overline{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$.*

Proof. By Theorem 4.26 we have $d\log q_f(\alpha,\beta_f) = 0$ for all $\alpha \in (B^t)^e(\overline{\mathbf{Q}}_p) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$ if and only if $(\eta, [M^{k'}\omega_f])_{\mathbf{M}'_{Ig,H'}} = 0$ for all $\eta \in \mathrm{H}^0(\mathbf{M}'_{Ig,H'}, \Omega^1_{\mathbf{M}'_{Ig,H'}/\overline{\mathbf{F}}_p})$. But by Theorem 4.14 $[M^{k'}\omega_f]$ is in the unit root subspace of $\mathrm{H}^1_{dR}(\mathbf{M}'_{Ig,H'}/\overline{\mathbf{F}}_p)$, which has trivial intersection with the space of global differentials, which is a maximal isotropic subspace for the pairing $(,)_{\mathbf{M}'_{Ig,H'}}$. So if $(\eta, [M^{k'}\omega_f])_{\mathbf{M}'_{Ig,H'}} = 0$ for $\eta \in \mathrm{H}^0(\mathbf{M}'_{Ig,H'}, \Omega^1_{\mathbf{M}'_{Ig,H'}/\overline{\mathbf{F}}_p})$ then $[M^{k'}\omega_f] = 0$. The converse is clear. □

5 The Main Theorem

5.1 Proof of the main result

From Theorem 4.13, Lemma 4.25 and Proposition 4.28 we see that we have constructed a companion form $\omega_{f'}$ as a differential. Note that $\omega_{f'}$ has character (of $(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})^\times$)) $\omega_{\mathfrak{p}}^{-k}$ at $\mathfrak{p}$, and $\overline{\rho}_{\pi'} \cong \overline{\rho}_\pi$. Furthermore, it has character $\omega_{\mathfrak{p}_i}^{-k'}$ at $\mathfrak{p}_i$ for $\mathfrak{p}_i \neq \mathfrak{p}$, because the diamond operators at $\mathfrak{p}_i$ act trivially on a and thus on ω_A. We now reverse the process that associated ω_f to π, associating a mod p Hilbert modular form π' to $\omega_{f'}$. Firstly, by the Deligne-Serre lemma there is a characteristic zero differential $\omega_{F'}$ whose Hecke eigenvalues lift those of $\omega_{f'}$. This differential corresponds to an automorphic form π_2 on $G'(\mathbf{A_Q})$ such that ψ'_∞ is of weight 2, where $BC(\pi_2) = (\psi', \Pi')$. Put $\pi'_E = JL(\Pi')$. Note that $\pi'_E \cong (\pi'_E)^\vee \circ c$; if now η' is a character such that $\eta'^c/\eta' = \chi_{\pi'_E}$, we have

$$\begin{aligned}(\pi'_E \otimes \eta') \circ c &\cong (\pi'_E)^c \otimes \eta'^c \\ &\cong (\pi'_E)^\vee \otimes \eta'^c \\ &\cong \pi'_E \otimes \chi^{-1}_{\pi'_E} \otimes \eta'^c \\ &\cong \pi'_E \otimes \eta',\end{aligned}$$

so there exists π' on $\mathrm{GL}_2(\mathbf{A}_F)$ such that $BC_{E/F}(\pi') = \pi'_E \otimes \eta'$. Again, $(\eta')_\infty$ is trivial, and we choose η' so that $\eta'_{\mathfrak{p}}$ is trivial. We wish to check that π' is ordinary at all primes dividing p; this follows from

Theorem 5.1. *Let Π be a weight two Hilbert modular form of level $\mathfrak{n}p$ and character ϵ (a strict ray class character of conductor $\mathfrak{n}p$), and suppose it has character $\omega_{\mathfrak{p}_i}^j$ at $\mathfrak{p}_i$, with $0 < j < p-2$. Let the slope of Π be the p-adic valuation of the eigenvalue $a_{\mathfrak{p}_i}$ of $U_{\mathfrak{p}_i}$ on Π. Then:*

- *If Π has slope 0, then* $\overline{\rho}_\Pi|_{I_{\mathfrak{p}_i}} \cong \begin{pmatrix} \omega_{\mathfrak{p}_i}^{j+1} & * \\ 0 & 1 \end{pmatrix}$;
- *If Π has slope 1, then* $\overline{\rho}_\Pi|_{I_{\mathfrak{p}_i}} \cong \begin{pmatrix} \omega_{\mathfrak{p}_i} & * \\ 0 & \omega_{\mathfrak{p}_i}^j \end{pmatrix}$;
- *If Π has slope in the interval(0,1), then $\overline{\rho}_\Pi|_{I_{\mathfrak{p}_i}} \cong \omega_2^{1+j} \oplus \omega_2^{p(1+j)}$, where ω_2 is a fundamental character of niveau 2 associated to $\mathfrak{p}_i$.*

Proof. The proof is very similar to that of Proposition 6.17 of [Sav04]. In fact, the proof, being purely local (and recall that we are in the case where p is totally split in F), is identical once one has a generalisation of the results of [Sai97] to the Hilbert case, for which see Theorem 1 of [Sai03] ☐

We now repeat the above arguments at all other primes of F dividing p, until we obtain a weight 2 level $\mathfrak{n}p$ Hilbert modular form π' with character $\omega_{\mathfrak{p}_i}^{-k'}$ at all $\mathfrak{p}_i|p$, with $\overline{\rho}_{\pi'}|_{\mathrm{Gal}(\overline{E}/E)} \cong \overline{\rho}_\pi|_{\mathrm{Gal}(\overline{E}/E)}$.

This does not, of course, guarantee that we have $\overline{\rho}_{\pi'} \cong \overline{\rho}_\pi$. However:

Theorem 5.2. *There exists a weight* 2 *level* $\mathfrak{n}p$ *Hilbert modular form* π' *with character* $\omega_{\mathfrak{p}_i}^{-k'}$ *at all* $\mathfrak{p}_i|p$ *such that the representation* $\overline{\rho}_{\pi'} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *satisfies* $\overline{\rho}_{\pi'} \cong \overline{\rho}_{\pi}$.

Proof. Suppose not. For any imaginary quadratic extension $K/\mathbf{Q}$ as above, and CM extension $E = FK$ of F, we can construct a π'_K as above, satisfying $\overline{\rho}_{\pi'_K}|_{\mathrm{Gal}(\overline{E}/E)} \cong \overline{\rho}_f|_{\mathrm{Gal}(\overline{E}/E)}$. Since there are only finitely many mod p weight 2 level $\mathfrak{n}p$ Hilbert modular forms, there are only a finite number of "candidate" automorphic forms π_i. For each π_i there must (by the Cebotarev density theorem and our assumption that there is no such π' with $\overline{\rho}_{\pi'} \cong \overline{\rho}_{\pi}$) be infinitely many places at which π_i is unramified principal series with the "wrong" characters mod p. In particular, we may choose such a place v_i for each π_i; then choosing $K/\mathbf{Q}$ to split at the places of $\mathbf{Q}$ lying below all of the v_i gives a contradiction. $\square$

Theorem (Theorem A). *Let* F *be a totally real field in which an odd prime* p *splits completely. Let* π *be a mod* p *Hilbert modular form of parallel weight* $2 < k < p$ *and level* $\mathfrak{n}$*, with* $\mathfrak{n}$ *coprime to* p*. Suppose that* π *is ordinary at all primes* $\mathfrak{p}|p$*, and that the mod* p *representation* $\overline{\rho}_{\pi} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *is irreducible and is tamely ramified at all primes* $\mathfrak{p}|p$*. Then there is a companion form* π' *of parallel weight* $k' = p + 1 - k$ *and level* $\mathfrak{n}$ *satisfying* $\overline{\rho}_{\pi'} \cong \overline{\rho}_{\pi} \otimes \chi^{k'-1}$.

Proof. Firstly, we deal with the case where $[F(\zeta_p) : F] = 2$ and $\overline{\rho}_{\pi}|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ is reducible. In this case we can directly construct a companion form, in a similar fashion to [Wie04]. Since $\overline{\rho}_{\pi}|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ is reducible, $\overline{\rho}$ is induced from a character ψ on $\mathrm{Gal}(\overline{F}/F(\zeta_p))$. As in Lemma 2 of [Wie04] we take the Teichmuller lift $\widetilde{\psi} : \mathrm{Gal}(\overline{F}/F(\zeta_p)) \to \mathcal{O}^{\times}_{F[\zeta_p]}$, so that $\widetilde{\rho} := \mathrm{Ind}_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}^{\mathrm{Gal}(\overline{F}/F)} \widetilde{\psi}$ is an odd (as $p > 2$) lift of $\overline{\rho}$, corresponding to a modular form of weight 1. By hypothesis $\overline{\rho}|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ is the direct sum of two characters, one of which is unramified; but then it is easy to see that $\widetilde{\psi}^2$ must be unramified, which yields $k - 1 = p - 1$ or $k - 1 = (p - 1)/2$. In the first case, the weight one form is unramified principal series at p, and is thus the required companion form; and in the second case we twist with the quadratic character whose restriction to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$ is trivial, and then use Hida theory to move to the required companion form in weight $(p + 1)/2$.

In the general case, we must prove that the mod p, level $\mathfrak{n}p$ weight 2 modular form π' constructed in Theorem 5.2 is also of weight k' and level $\mathfrak{n}$. But again, it follows from the Hida theory in [Wil88] that we have an ordinary form of weight k' and level $\mathfrak{n}p$, and it remains to check that this form cannot be new at any $\mathfrak{p}_i|p$. This is, however, easy; as the character at $\mathfrak{p}_i$ is unramified (because we are in weight k'), $\pi'_{\mathfrak{p}_i}$ would have to be special, which immediately contradicts the fact that it is ordinary, provided that $k' \neq 2$; but in this case we may use Theorem 6.2 of [Jar04]. $\square$

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